Robot Perception and Learning

Robot Kinematics

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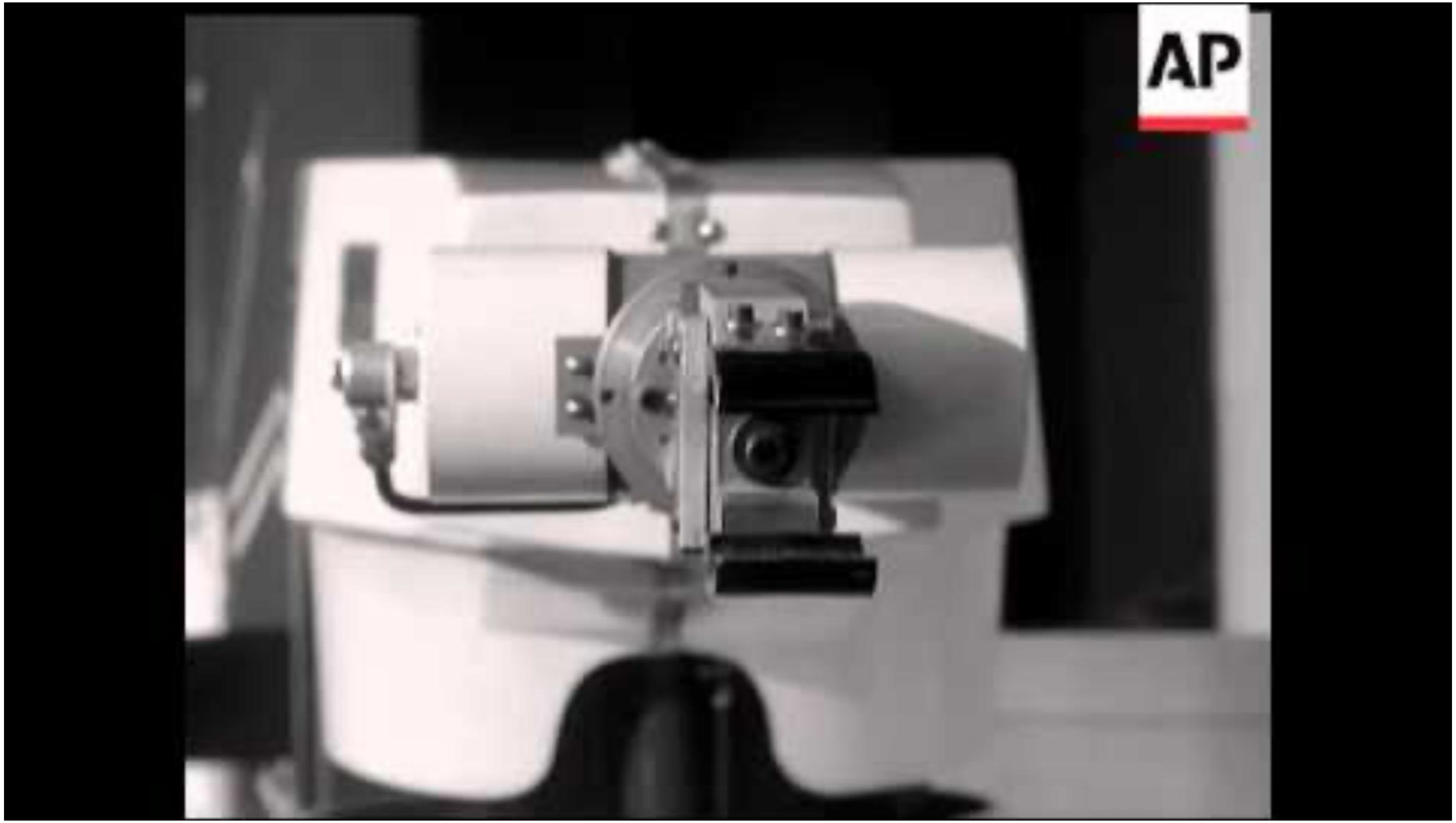
Fall 2025

We want a Robot that can See and Move

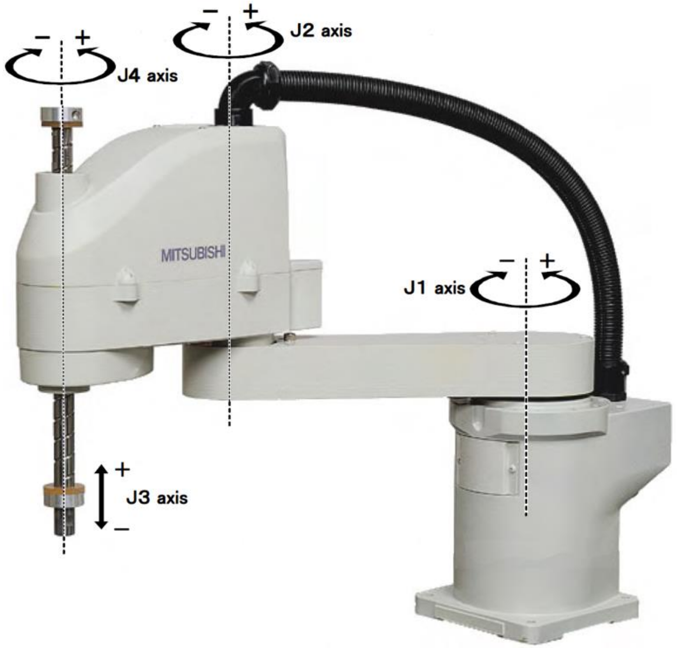


https://youtu.be/hxsWeVtb-JQ?si=1vDmFKZn8ZdZ9rZ2

The Stanford Arm. 1969

PR2 Robot. 2008

Clone Synthetic Hand. 2023

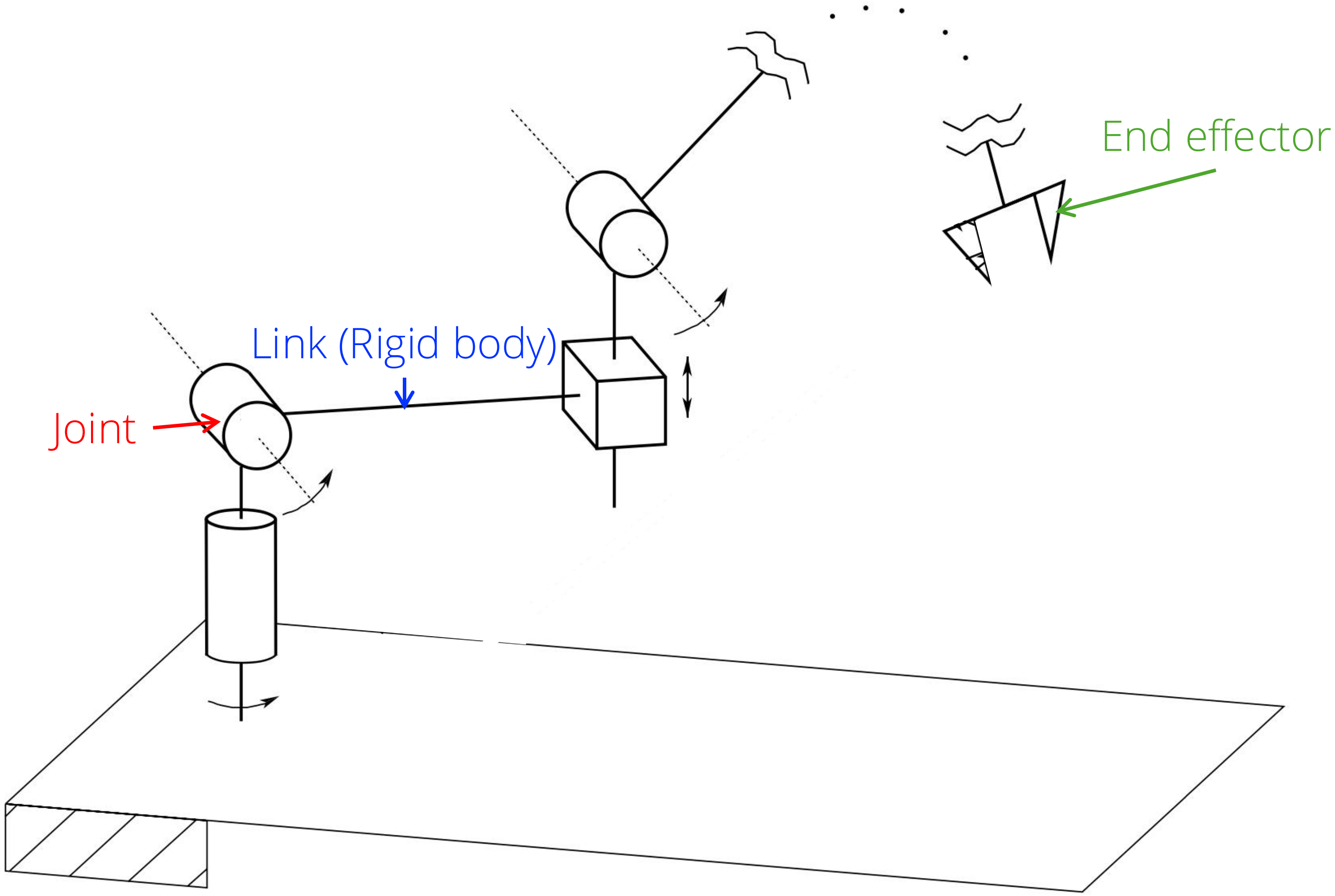


SCARA Robot. 1978

Franka Emika Panda Robot. 2017

Active entanglement enables stochastic, topological grasping. Becker et al

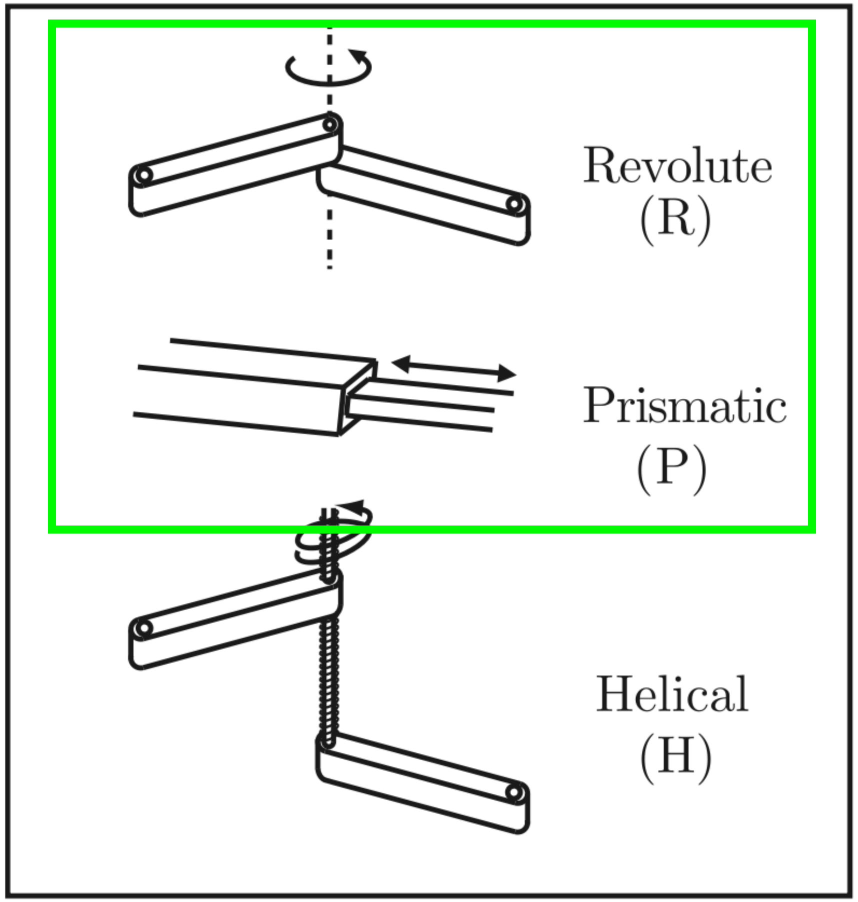
Anatomy of Rigid Multi-Link Robots

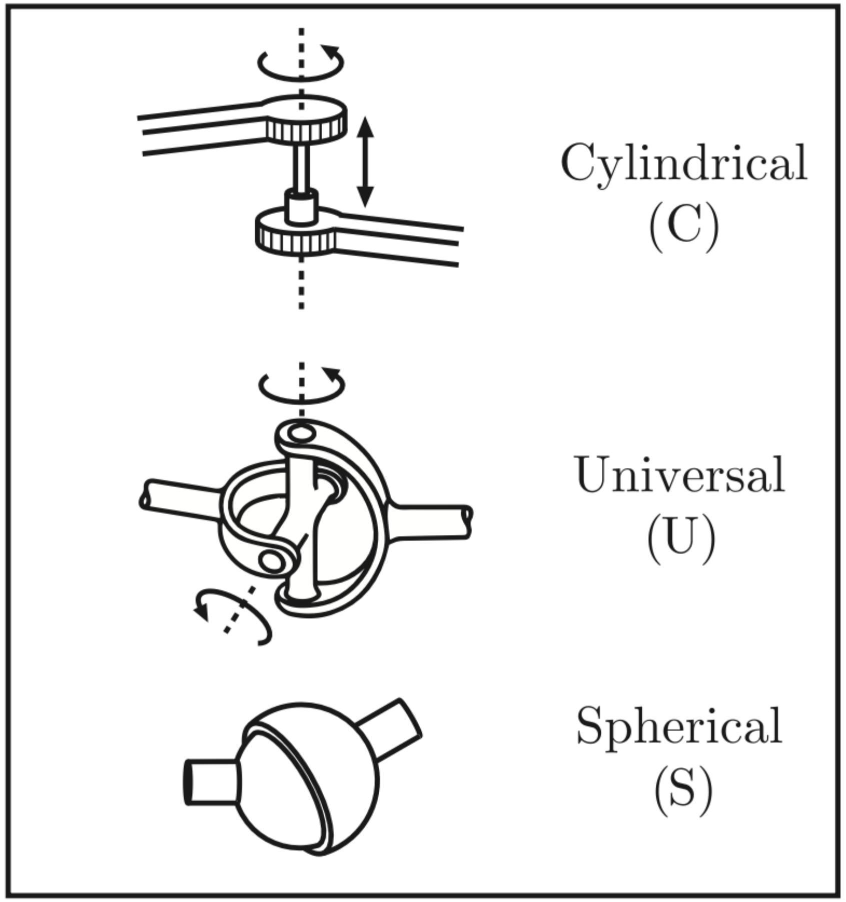


Robotics: Modeling, Planning and Control. Siciliano et al.

Typical Robot Joints

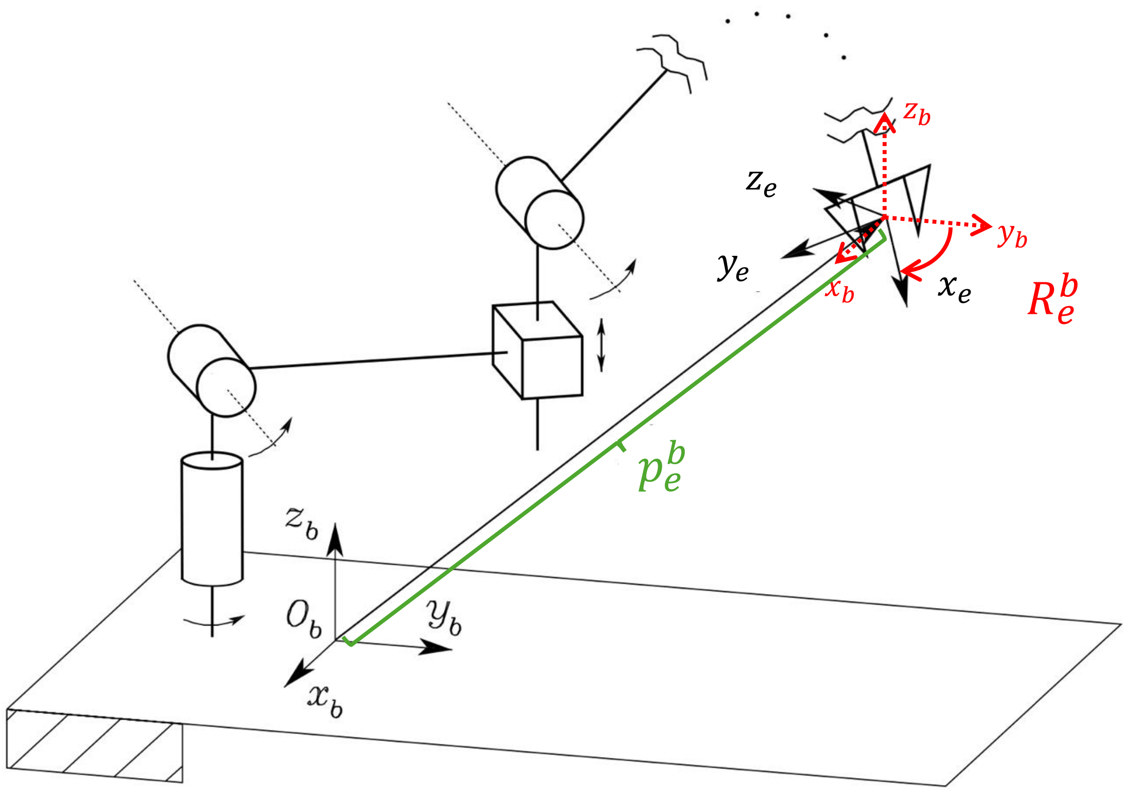
We will only discuss these two types of robot joints





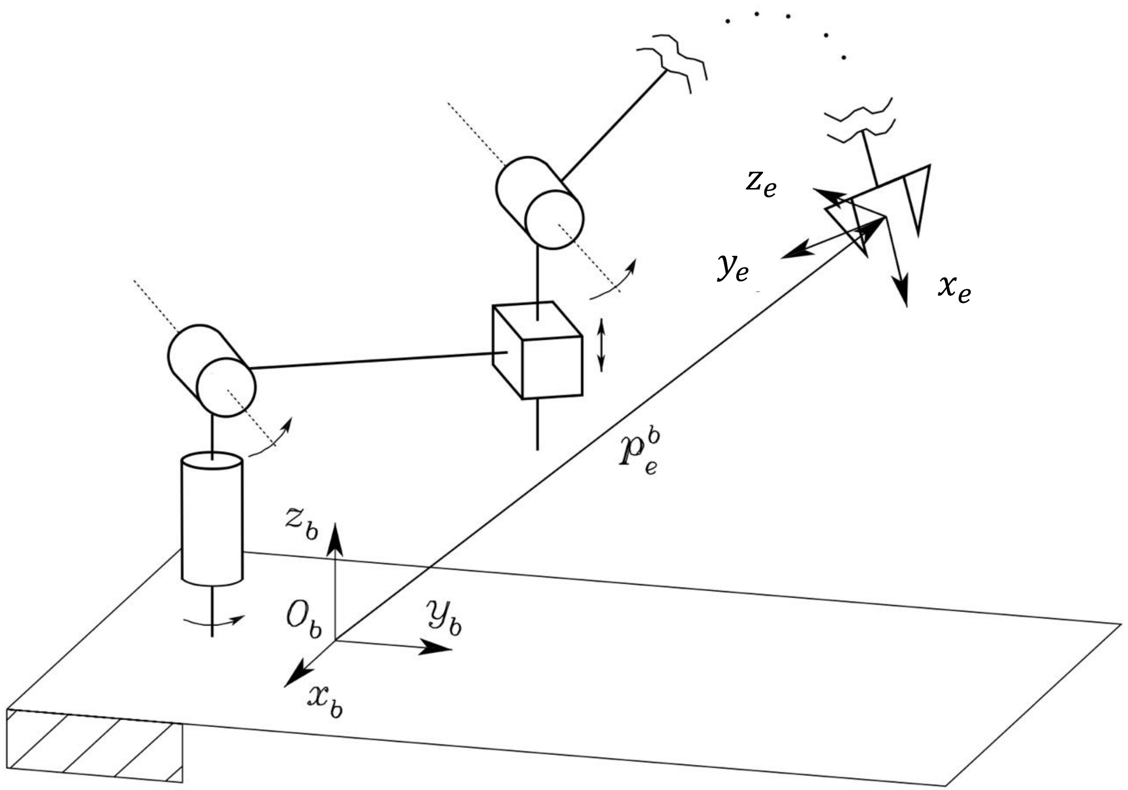
Where is the Robot?

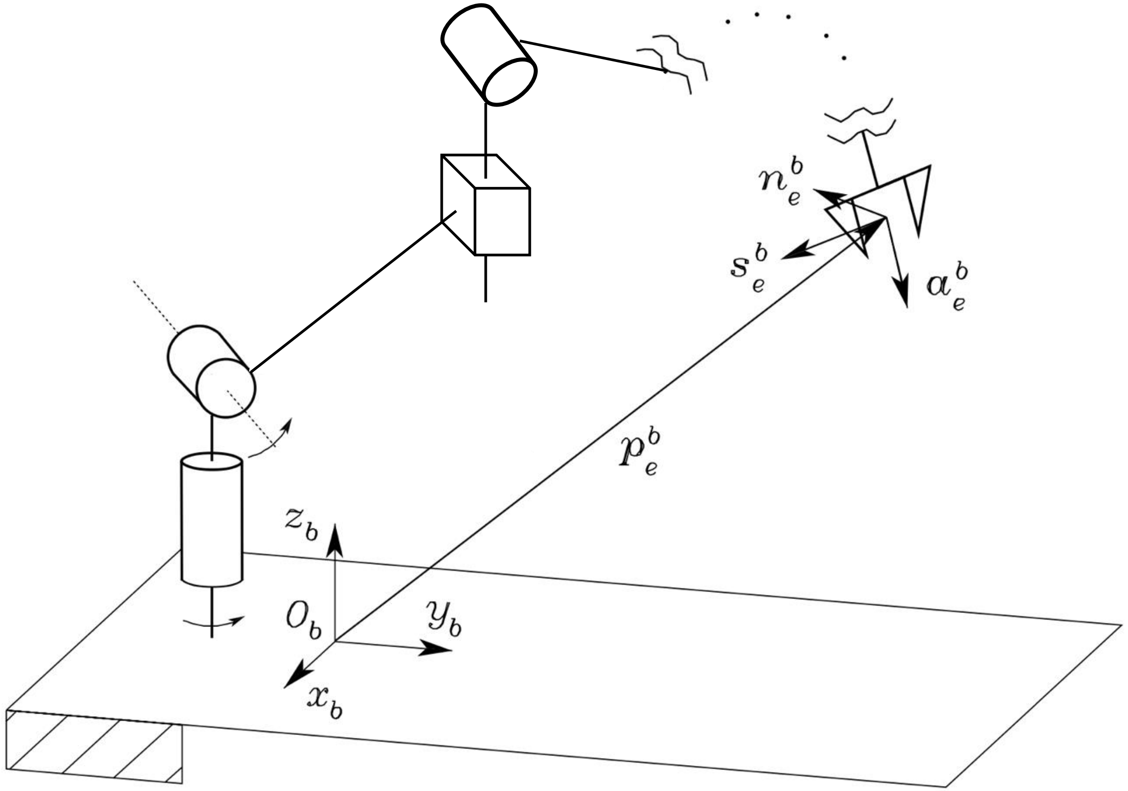
Idea: Specify the end-effector pose (location p_e^b and rotation R_e^b)



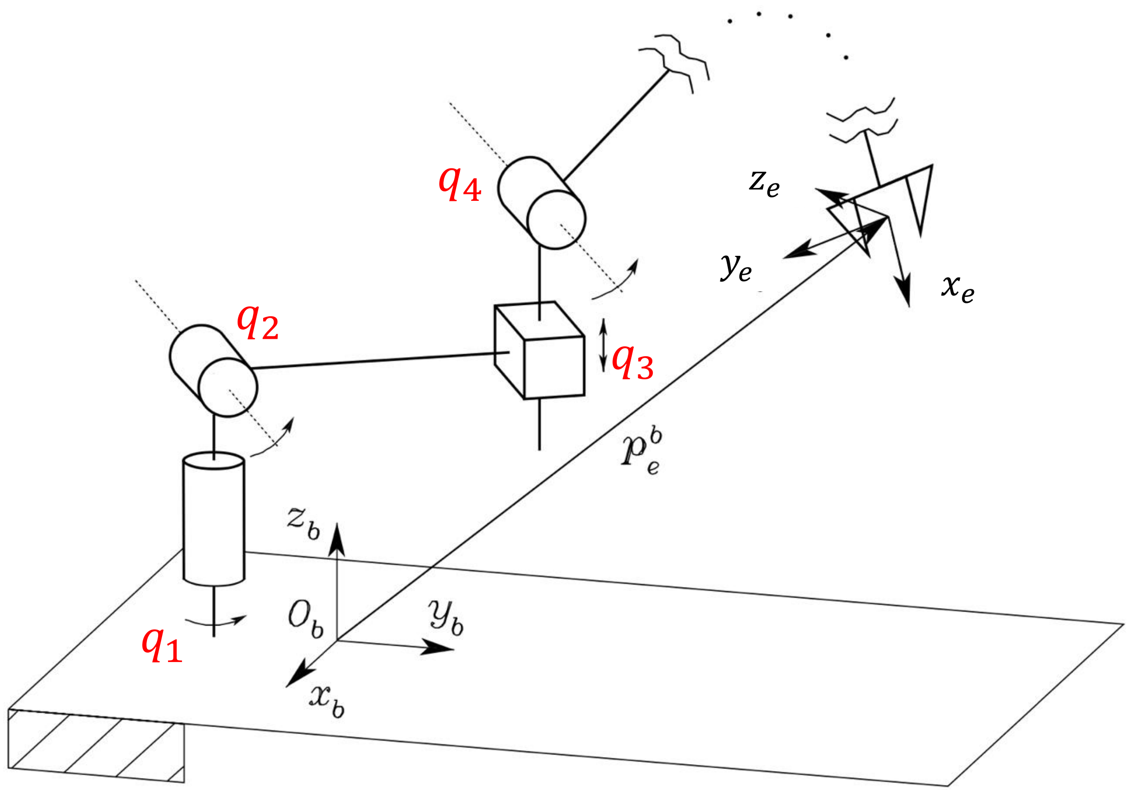
Where is the Robot?

However, these two **configurations** end up in the same end-effector pose





Configuration Space vs. Task (Operational) Space

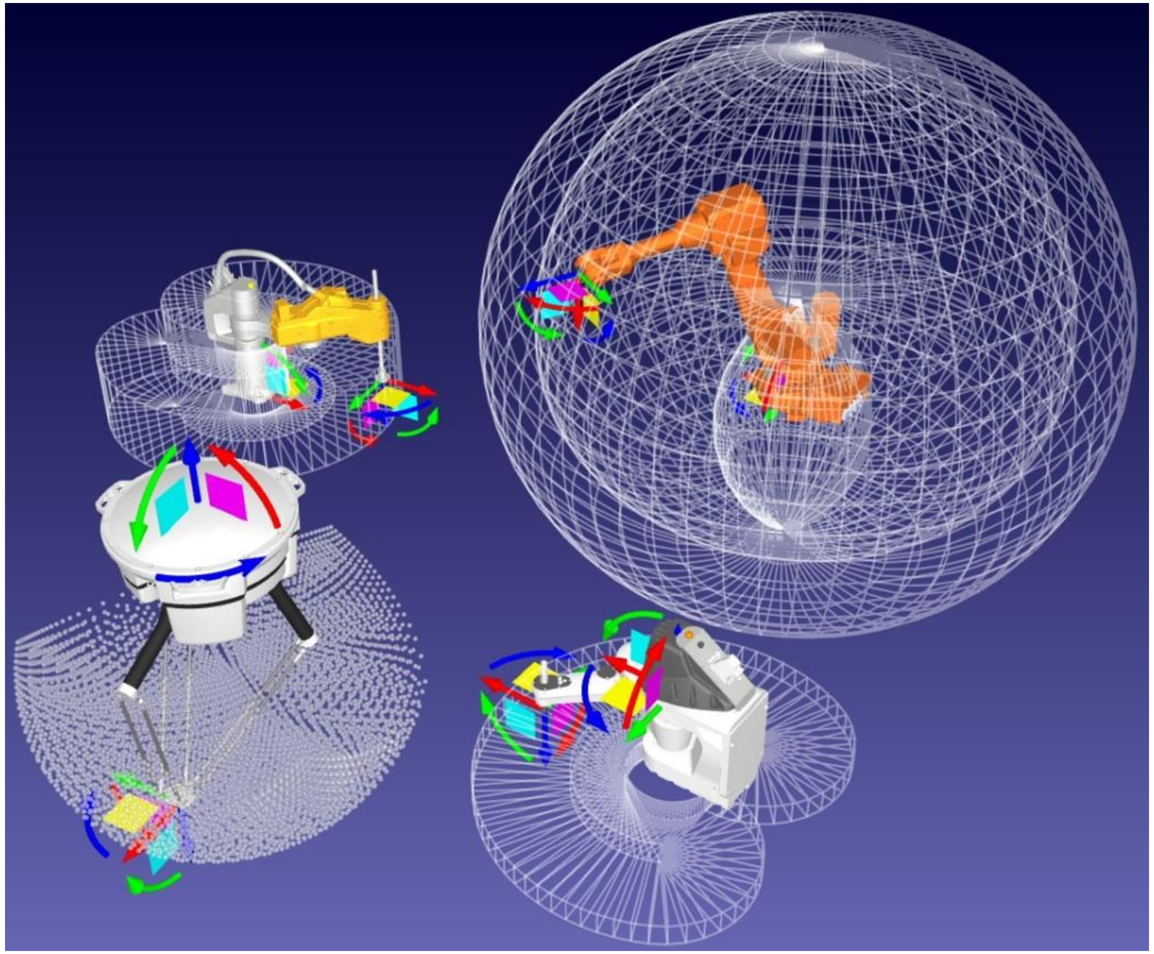


Configuration: a complete specification of the position of every point of the robot (e.g. angle of a revolute joint or displacement of a prismatic joint).

Configuration space (C-space): the n-dimensional space containing all possible configurations of the robot.

Task space: the space in which the robot's task is naturally expressed.

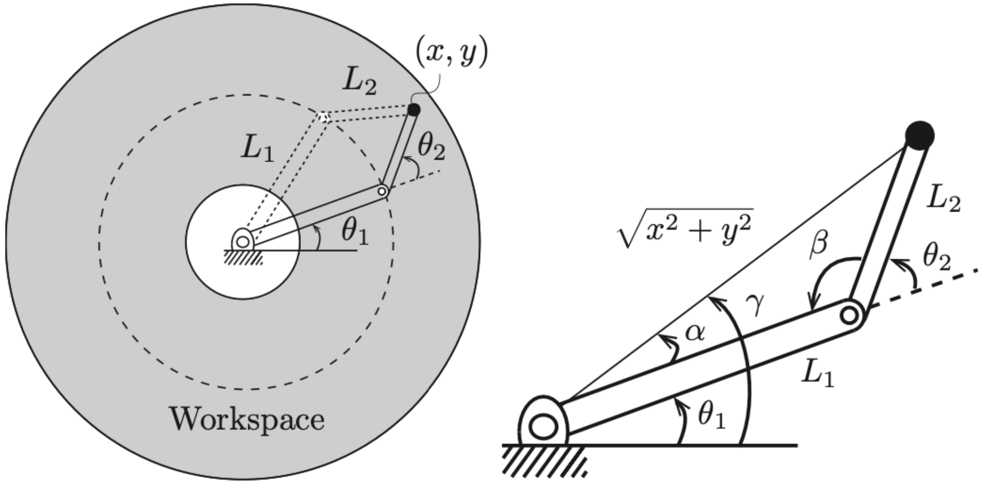
Task (Operational) Space vs. Work space



https://robodk.com/blog/robot-workspace-visualization/

Task space: the space in which the robot's task is naturally expressed.

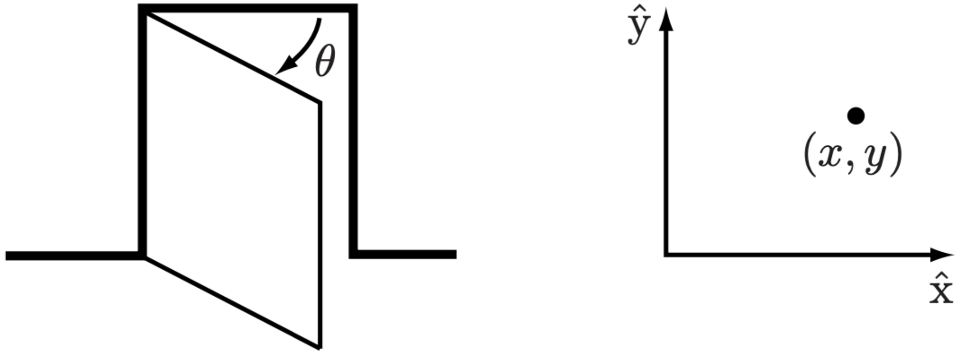
Work space: the space in which the robot's end-effector can reach.

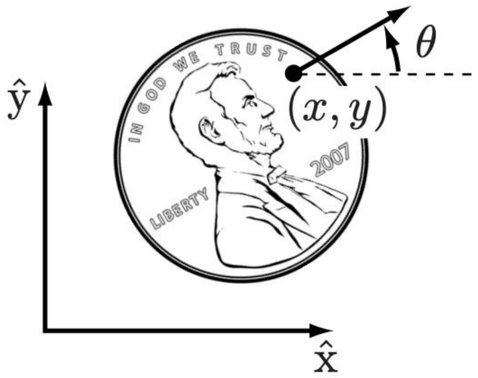


(a) A workspace, and lefty and righty configurations.

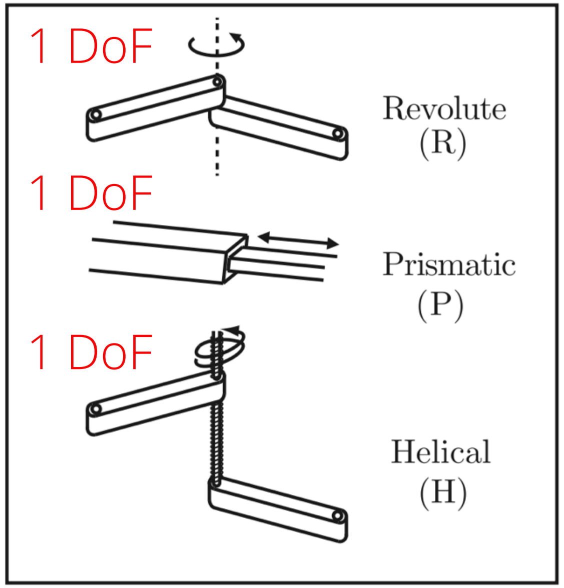
(b) Geometric solution.

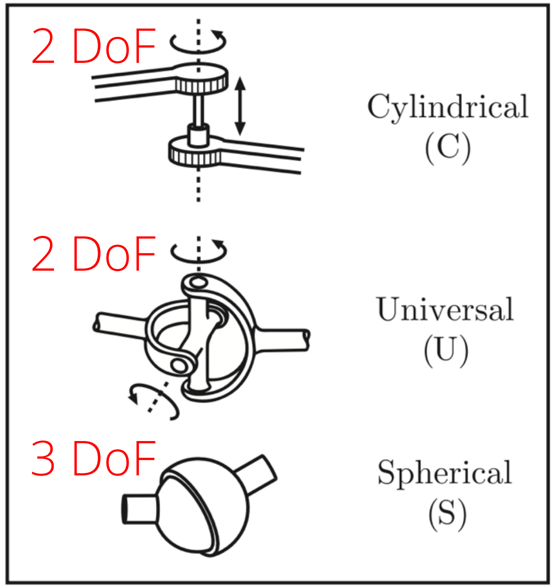
Degree of Freedom





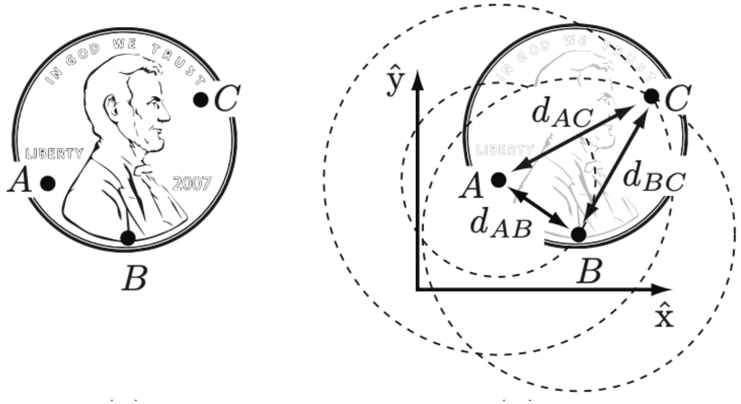
Degree of Freedom (DoF): The minimum number n of real-valued coordinates needed to represent the configuration.

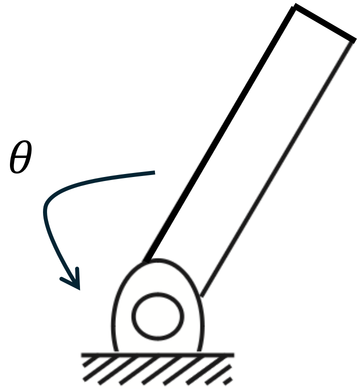


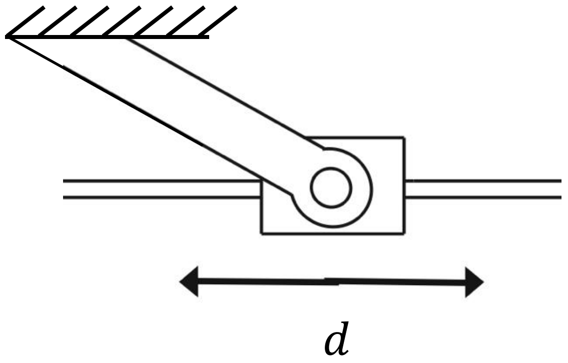


Degree of Freedom

Degree of Freedom: sum of freedoms of the bodies - #. of independent constraints
or
#. of variables - #. of independent equations







#. of variables = 2 (coordinates) * 3 (points)
#. of independent equations = 3 (d_{AC}, d_{AB}, d_{BC})
DoF = 3 (x, y, θ)

#. of variables = 6 ($X, Y, Z, pitch, roll, yaw$)
#. of independent equations = 5
DoF = 1 (θ)

#. of variables = 6 ($X, Y, Z, pitch, roll, yaw$)
#. of independent equations = 5
DoF = 1 (d)

Modern Robotics. K. M. Lynch and F. C. Park.

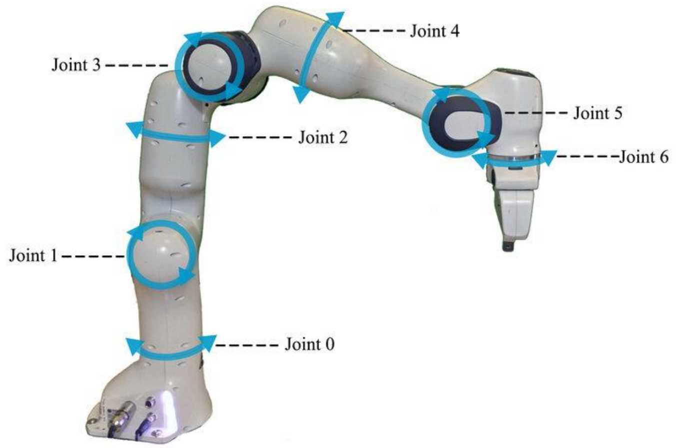
A robot is kinematically redundant if it has more DoF than the dimension of the task space, which provides more solutions to reach a pose.

Grübler's formula for the number of degrees of freedom of the robot is

#. links

$$\mathrm{dof} = \underbrace{m(\boxed{N} - 1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}}$$

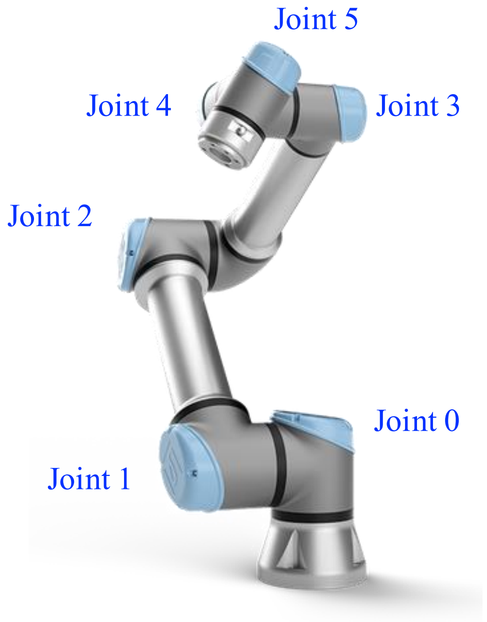
Franka Emika Panda Robot



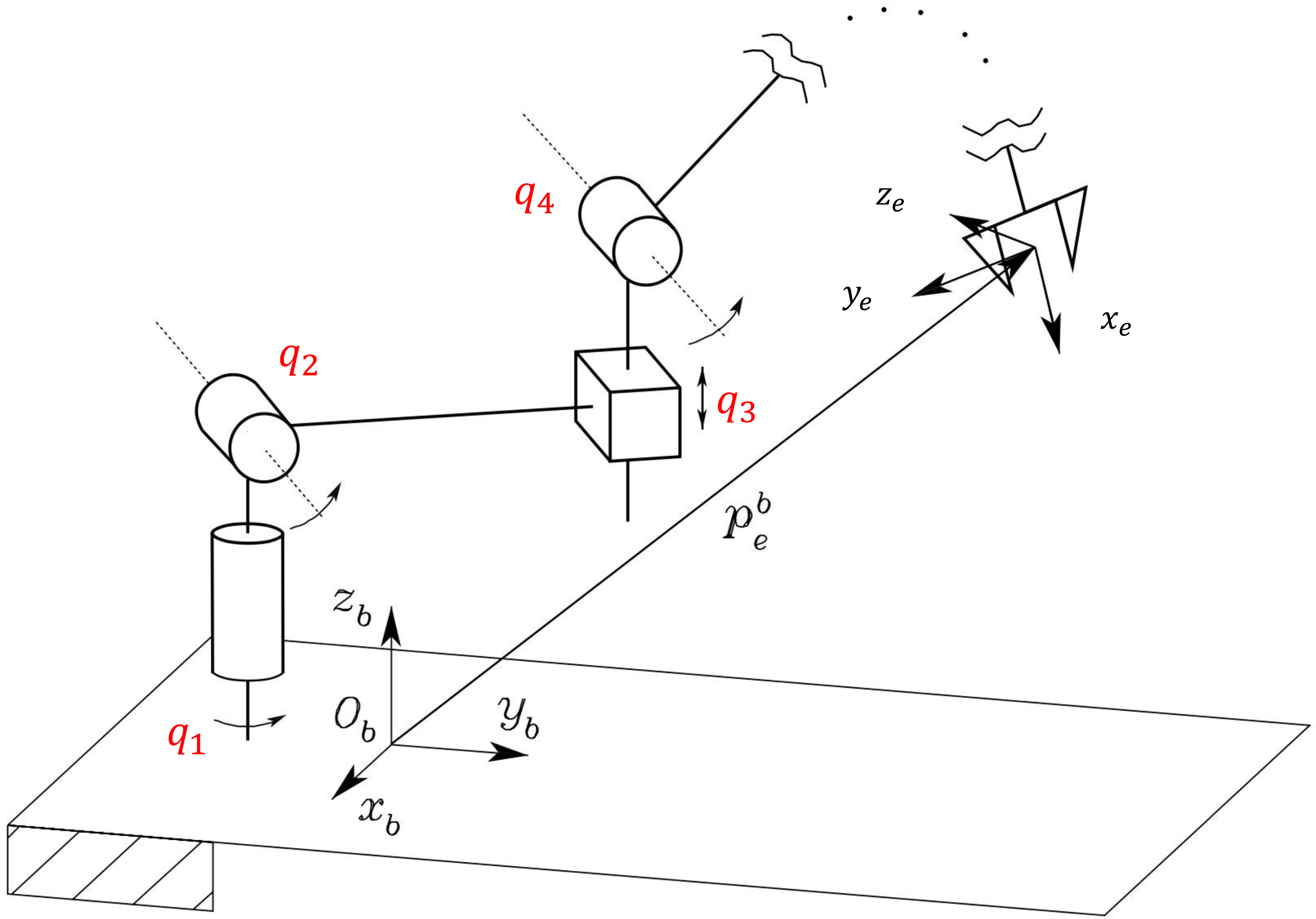
$$DoF = 6 * (8 - 1) - \sum_{i=0}^{6} 5 = 7$$

Why?

UR5E Robot

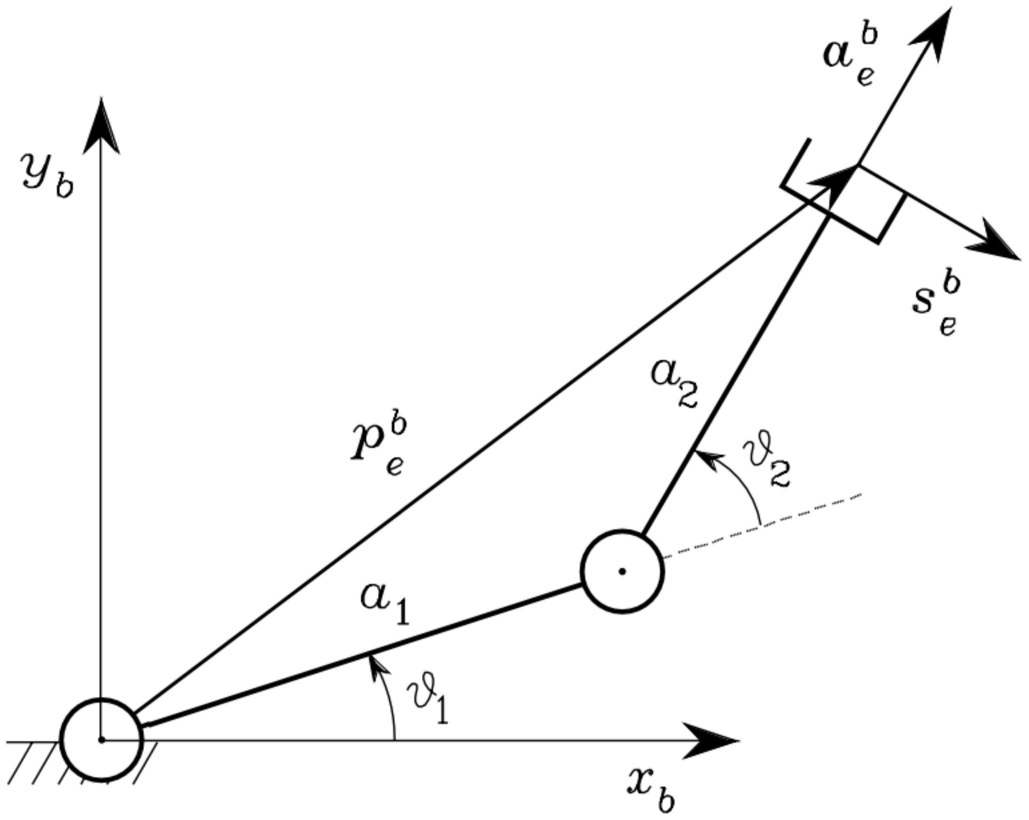


What is the Pose of a Robot's End Effector given Its Configuration?



Robotics: Modeling, Planning and Control. Siciliano et al.

Forward Kinematics: Calculate the Pose of a Robot End-Effector given the Configuration

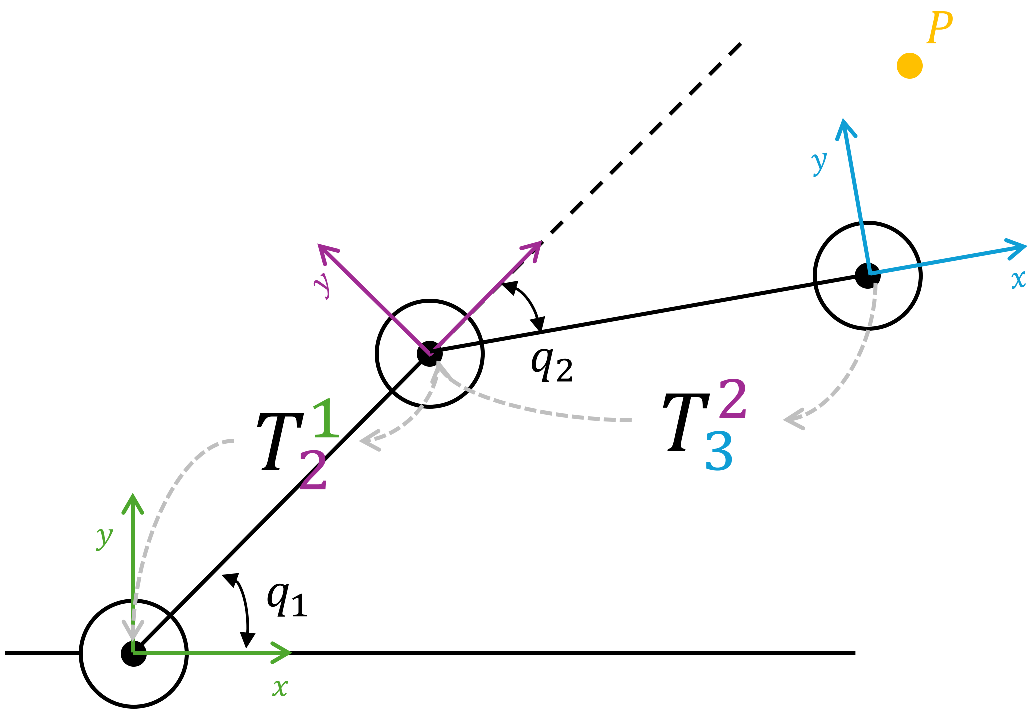


- We can calculate by Euclidean geometry, but is there any more general formulation?
- Let c_1 denote $\cos\vartheta_1$ and s_1 denote $sin\,\vartheta_1$

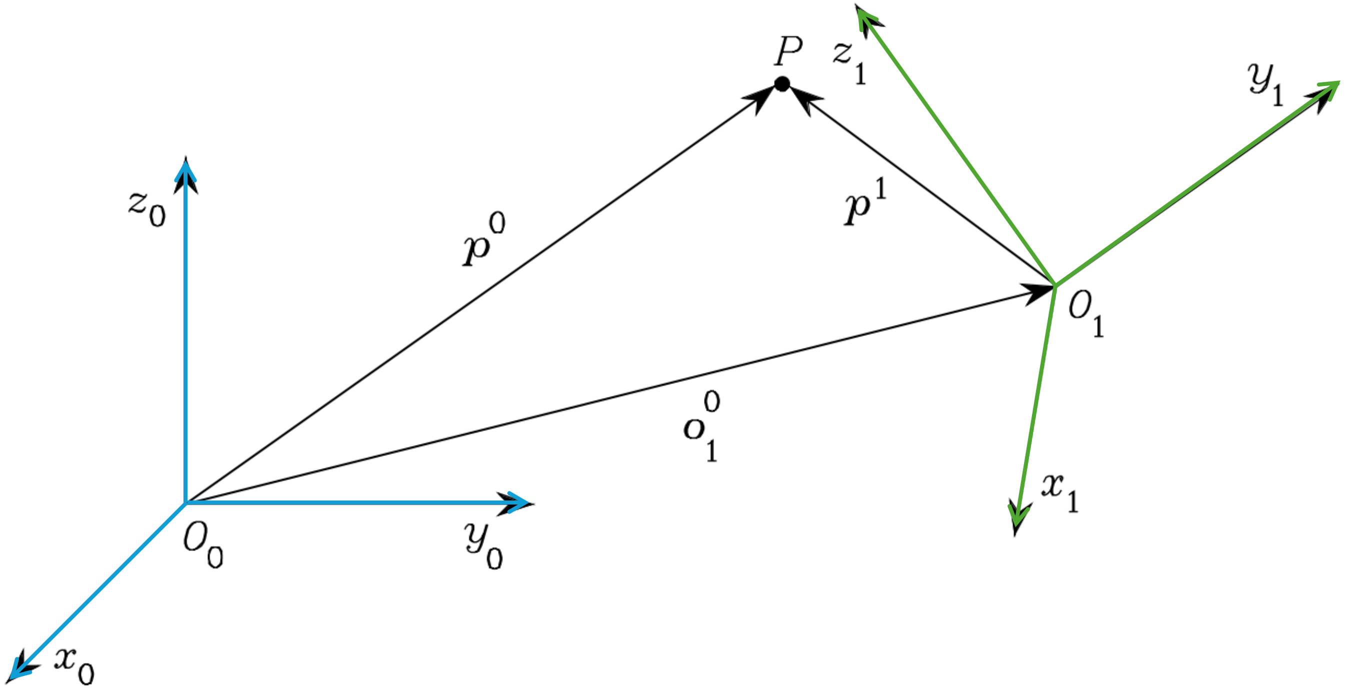
$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} 0 & s_{12} & c_{12} & a_1c_1 + a_2c_{12} \\ 0 & -c_{12} & s_{12} & a_1s_1 + a_2s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics: Calculate the Pose of a Robot End-Effector given the Configuration

- We can consider each joint applies coordinate transformation



Rigid-Body Transformation



The coordinate of point P in frame $\boldsymbol{O_0}$

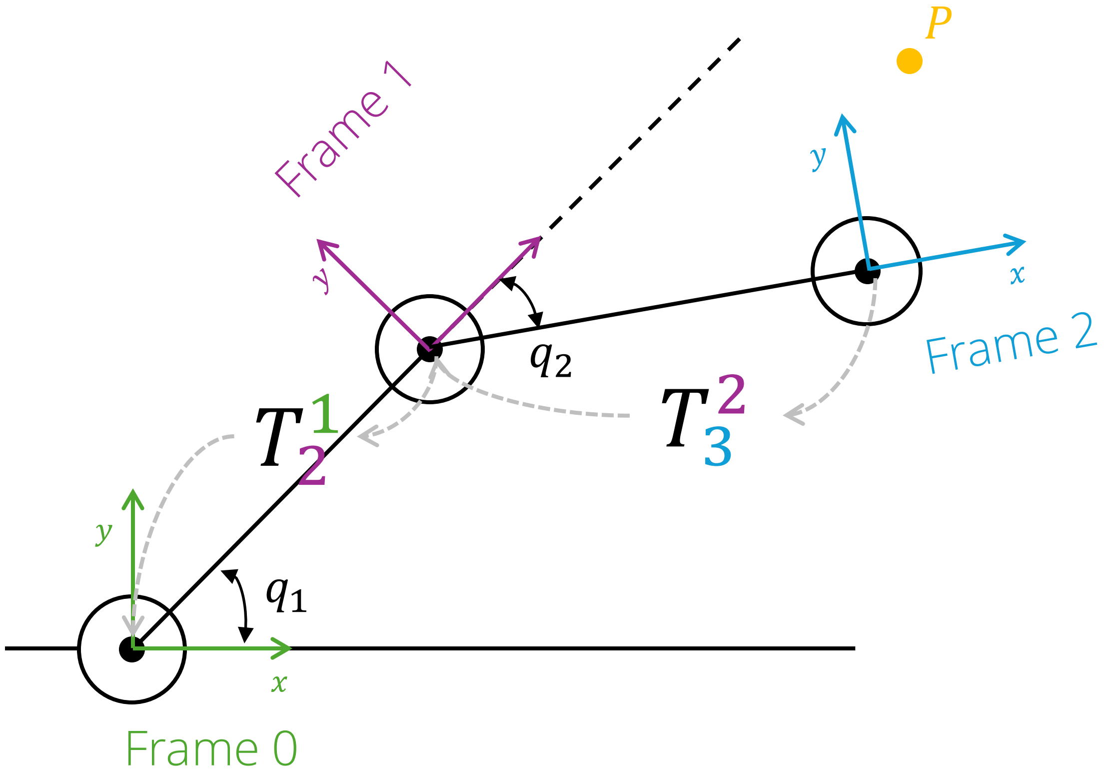
The coordinate of point P in frame $\boldsymbol{O_1}$

$$\begin{bmatrix} \boxed{\boldsymbol{p}^0} \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}} \begin{bmatrix} \boxed{\boldsymbol{p}^1} \\ 1 \end{bmatrix}$$

Spatial Transformation between frame $\boldsymbol{O_1}$ and $\boldsymbol{O_0}$

Forward Kinematics: Calculate the Pose of a Robot End-Effector given the Configuration

- We can consider each joint applies coordinate transformation

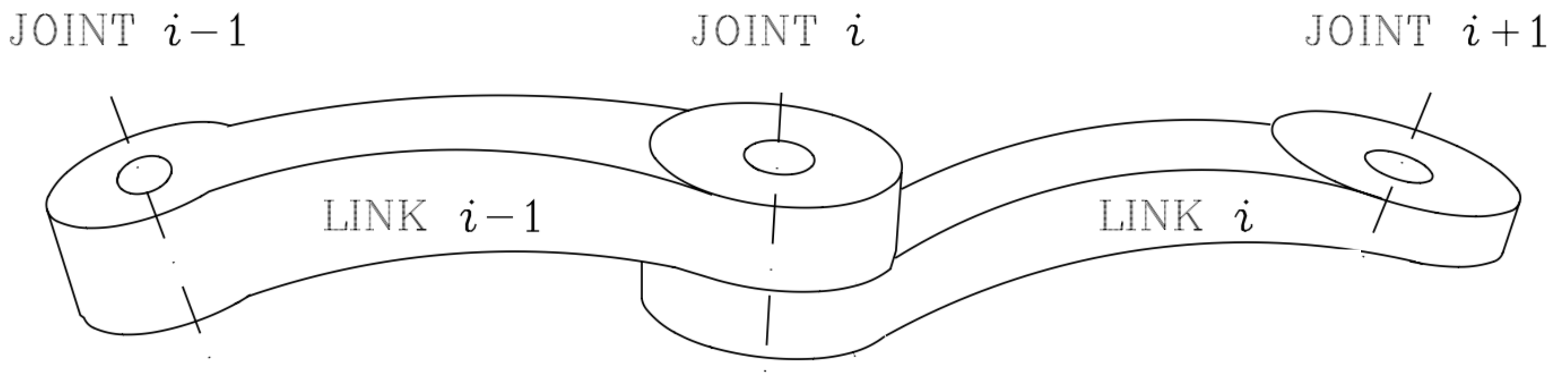


Coordinate transformation from the end-effector frame to Frame n

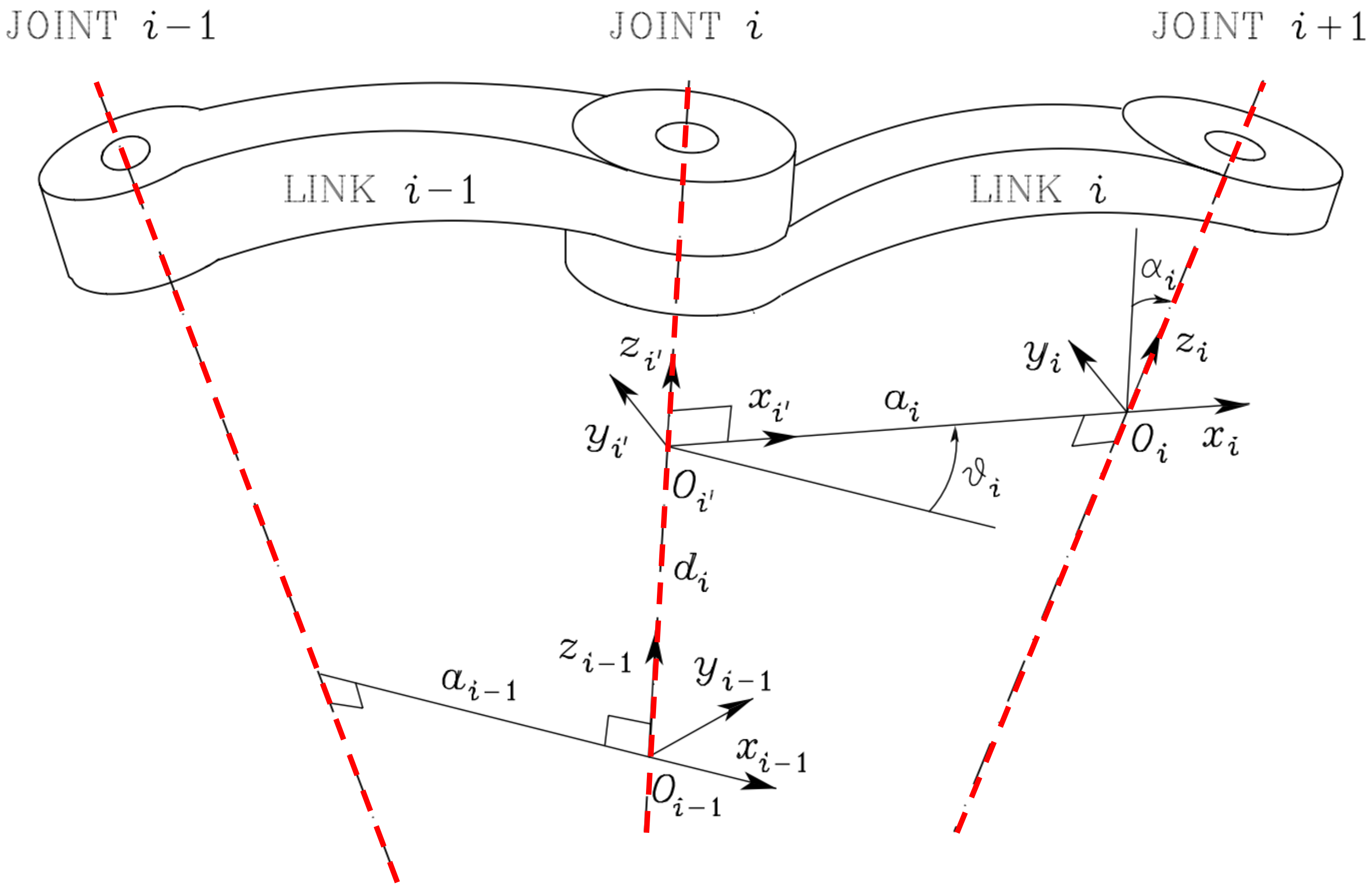
$$T_e^b(\boldsymbol{q}) = T_0^b T_1^0(q_1) \ldots T_n^{n-1}(q_n) T_e^n$$

Coordinate transformation from Frame 0 to the robot base frame

How to Determine Frames Attached to the Two Links?

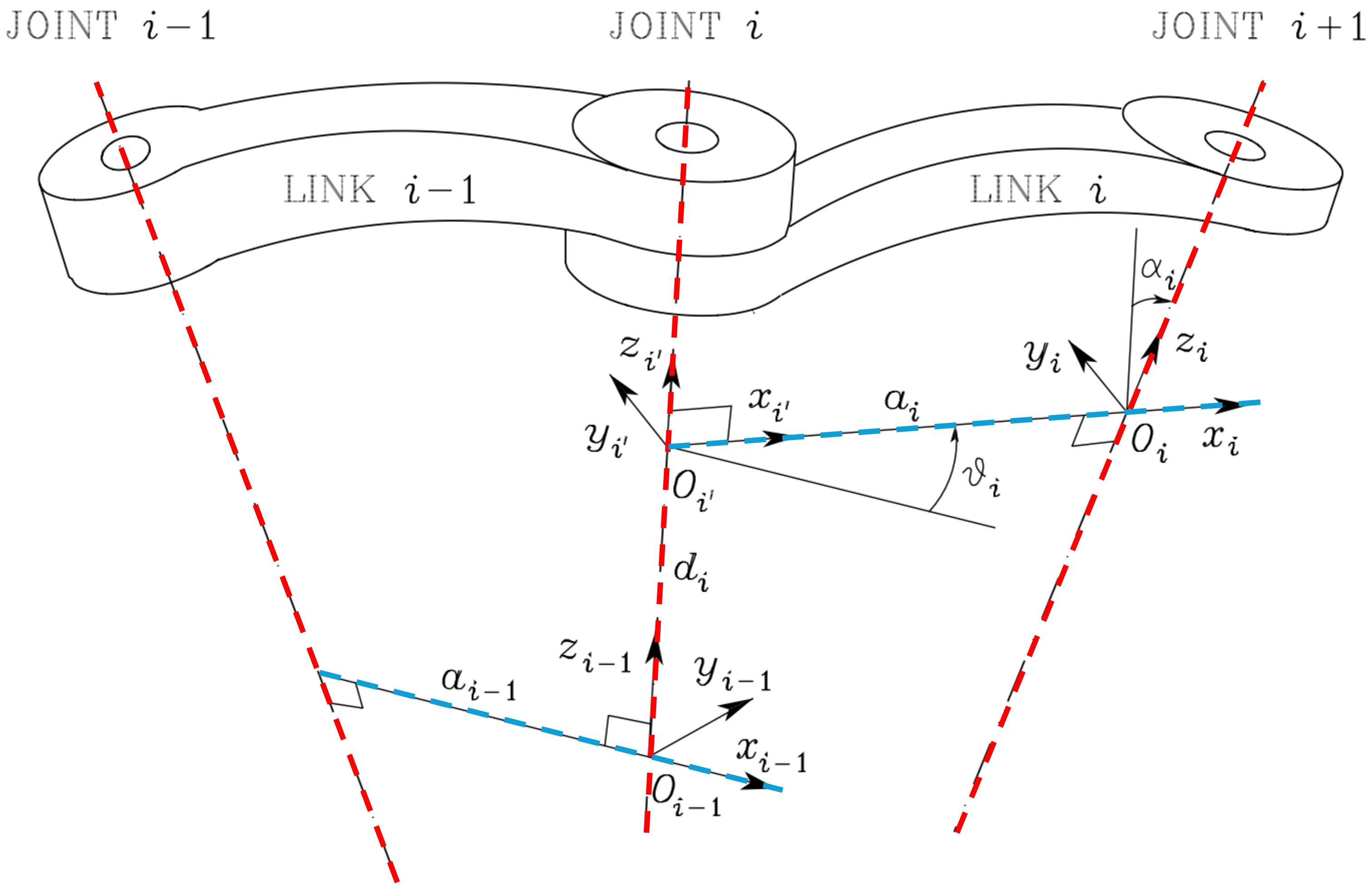


Decide Frames with Denavit–Hartenberg Convention



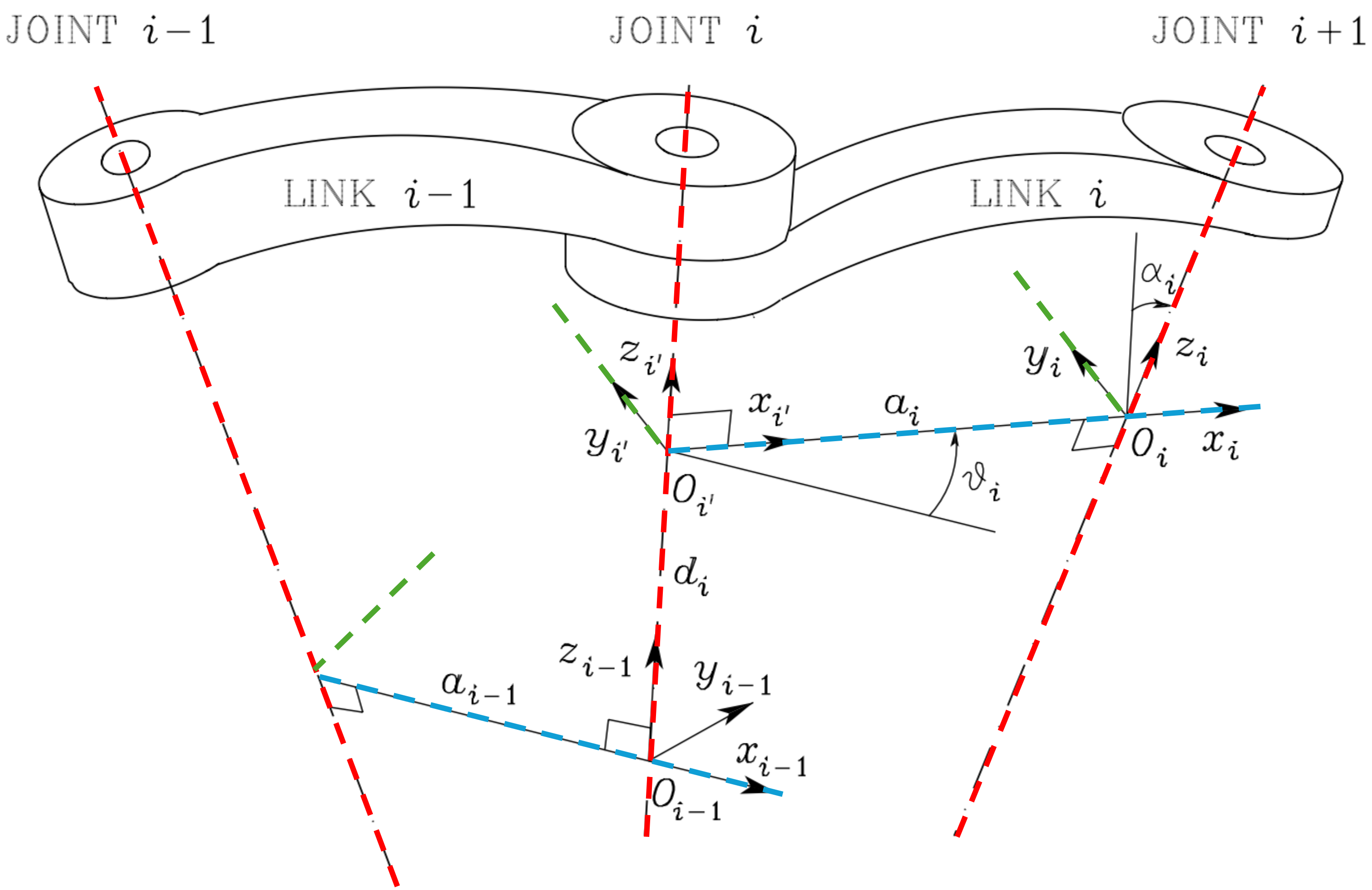
1. Choose axis z_i along the axis of Joint $i+1$

Decide Frames with Denavit–Hartenberg Convention



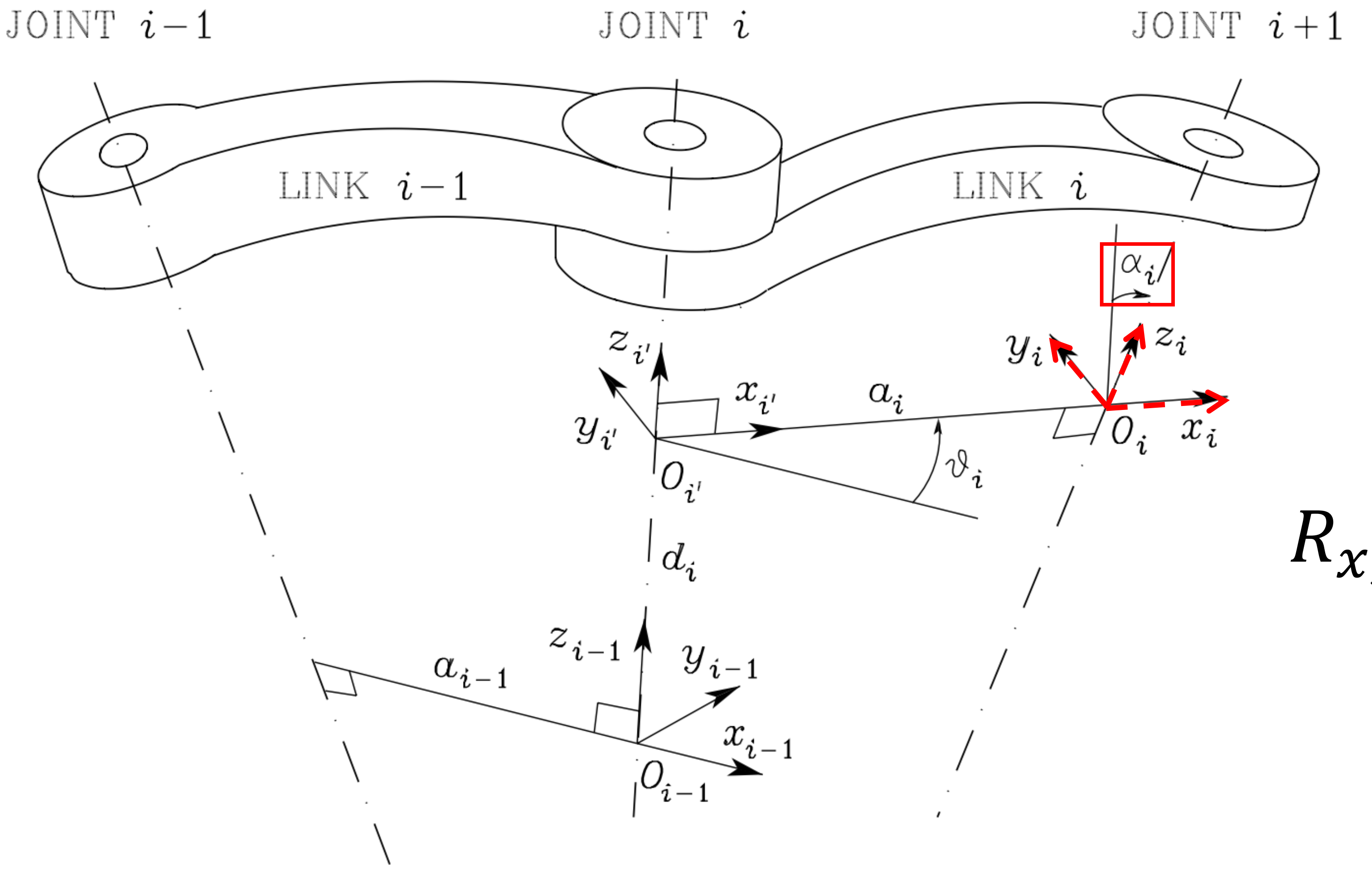
1. Choose axis z_i along the axis of Joint $i+1$
2. Choose axis x_i along the direction of the common normal (vector of minimum distance between axis z_i and z_{i-1})

Decide Frames with Denavit–Hartenberg Convention



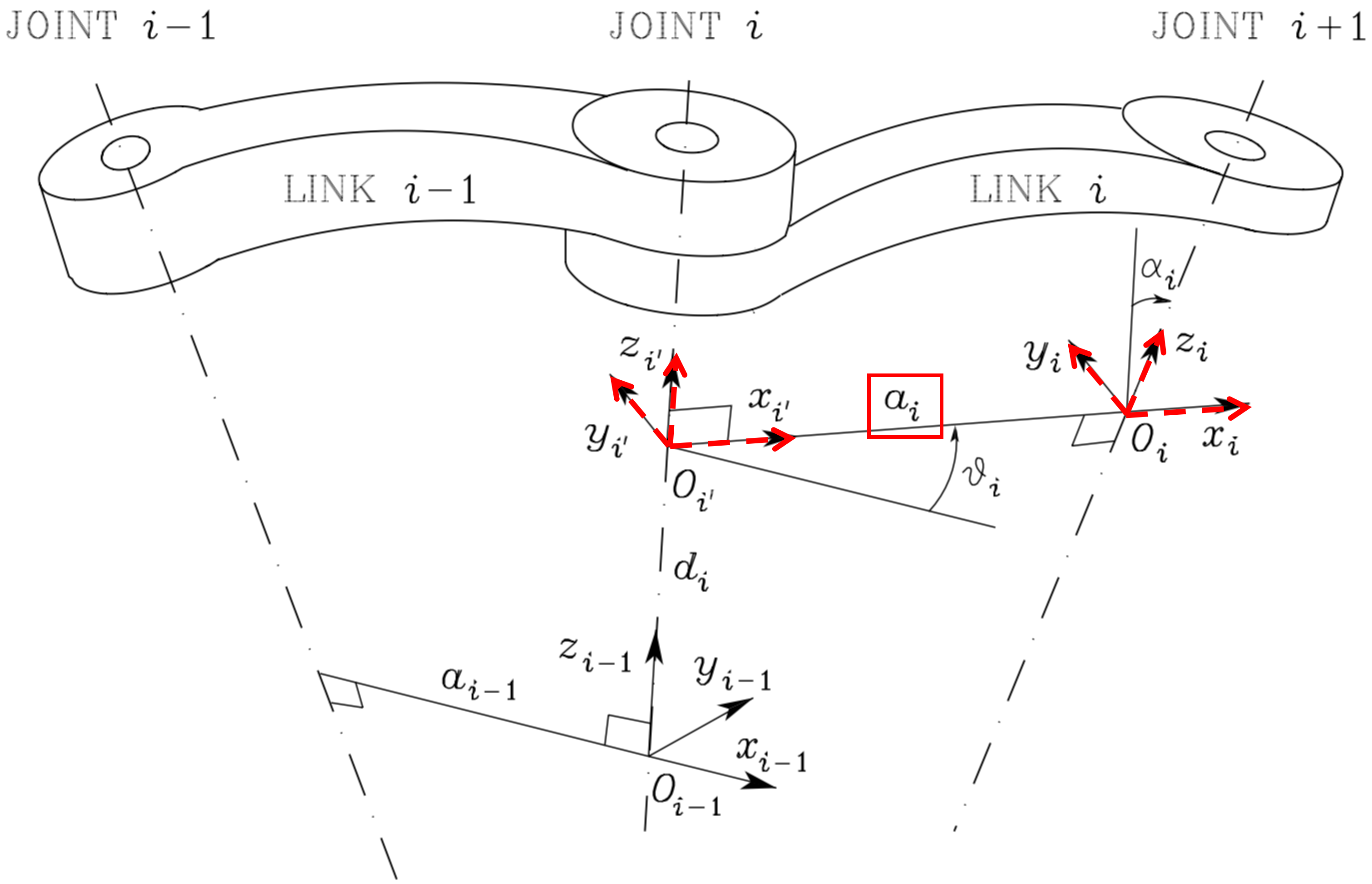
1. Choose axis z_i along the axis of Joint $i+1$
2. Choose axis x_i along the direction of the common normal (vector of minimum distance between axis z_i and z_{i-1})
3. Axis y_i is decided by right-hand rule

What are the Spatial Transformations under Denavit–Hartenberg Convention?



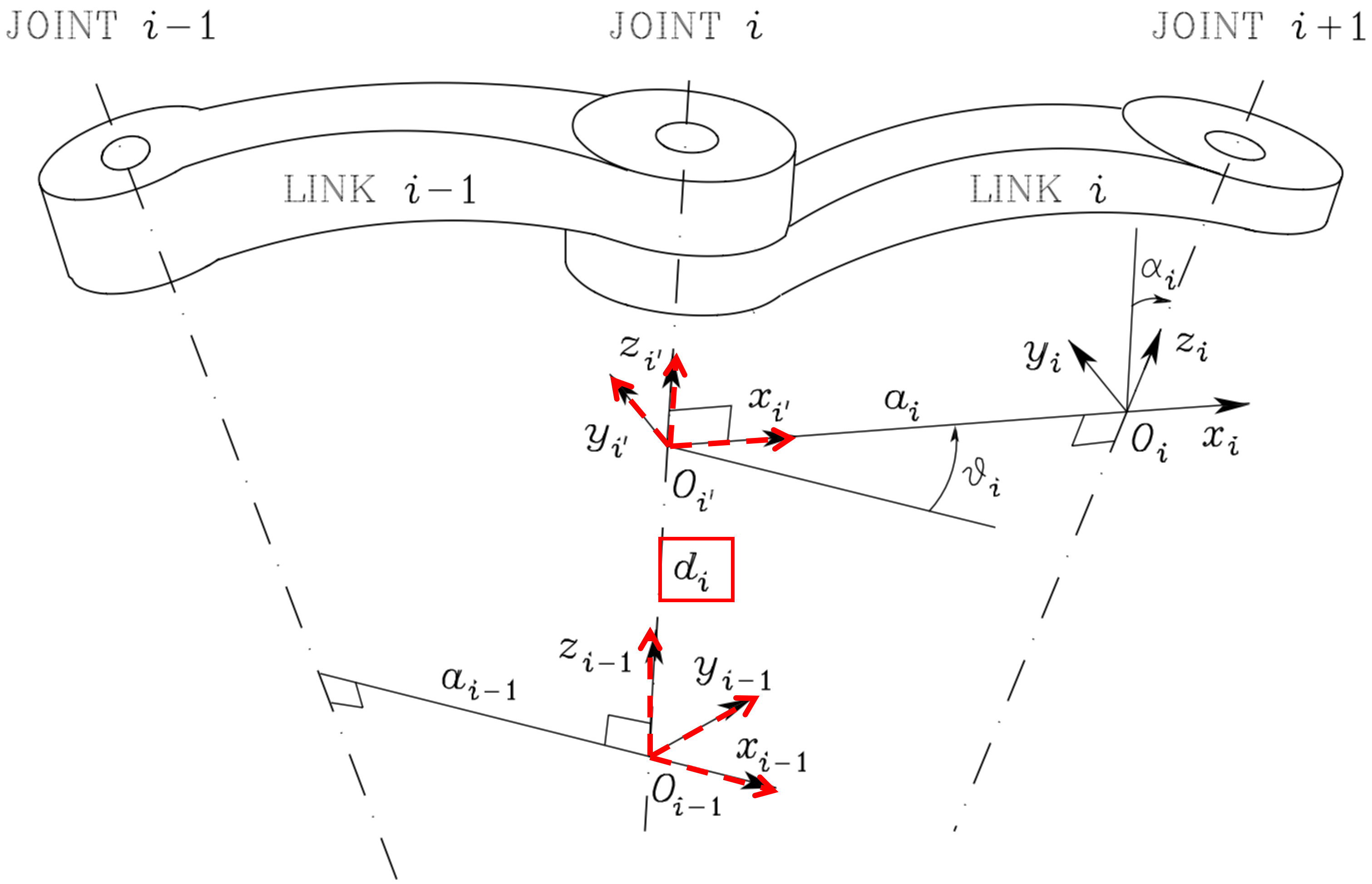
$$R_{x,\alpha_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the Spatial Transformations under Denavit–Hartenberg Convention?



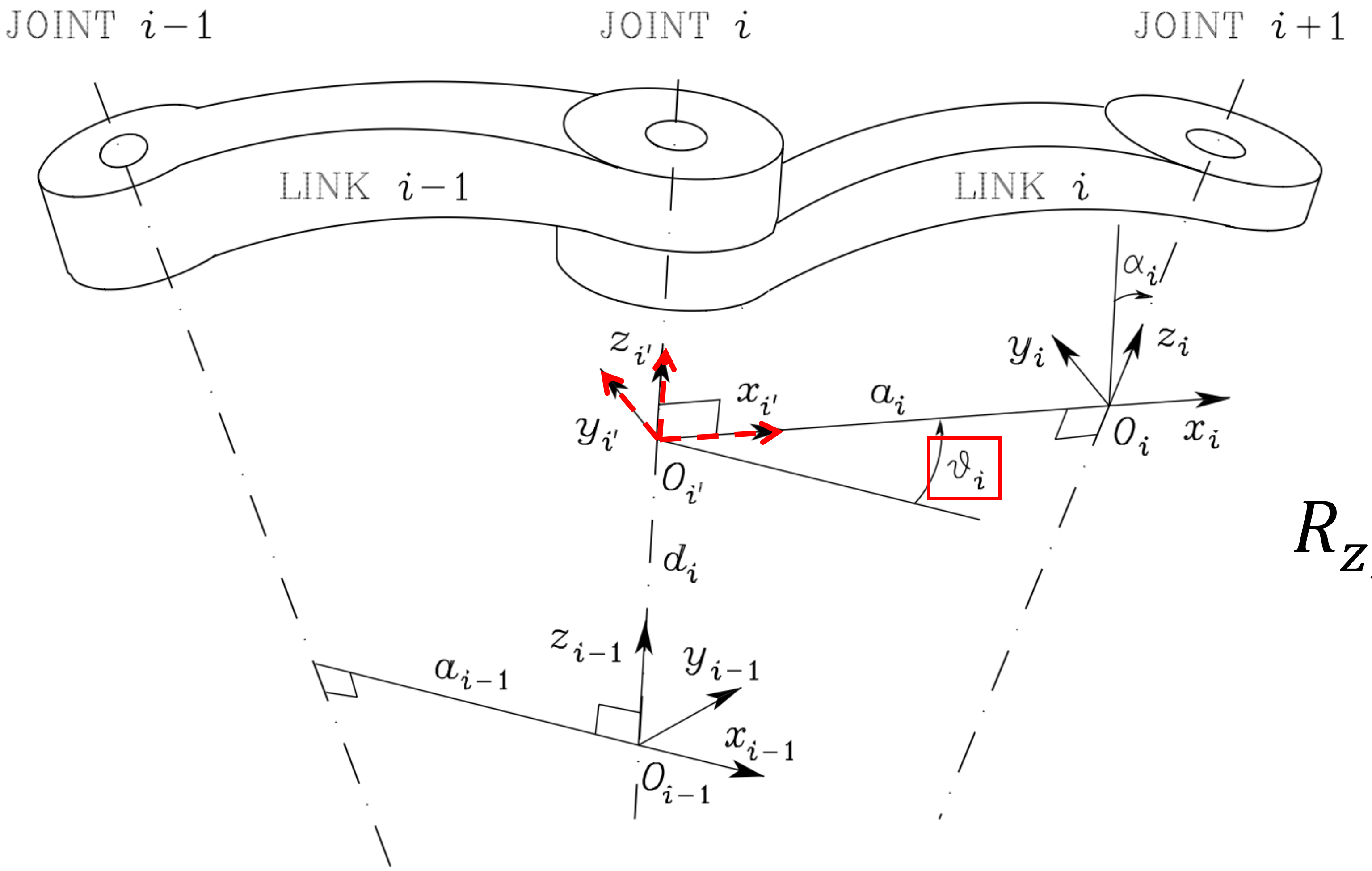
$$T_i^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the Spatial Transformations under Denavit–Hartenberg Convention?



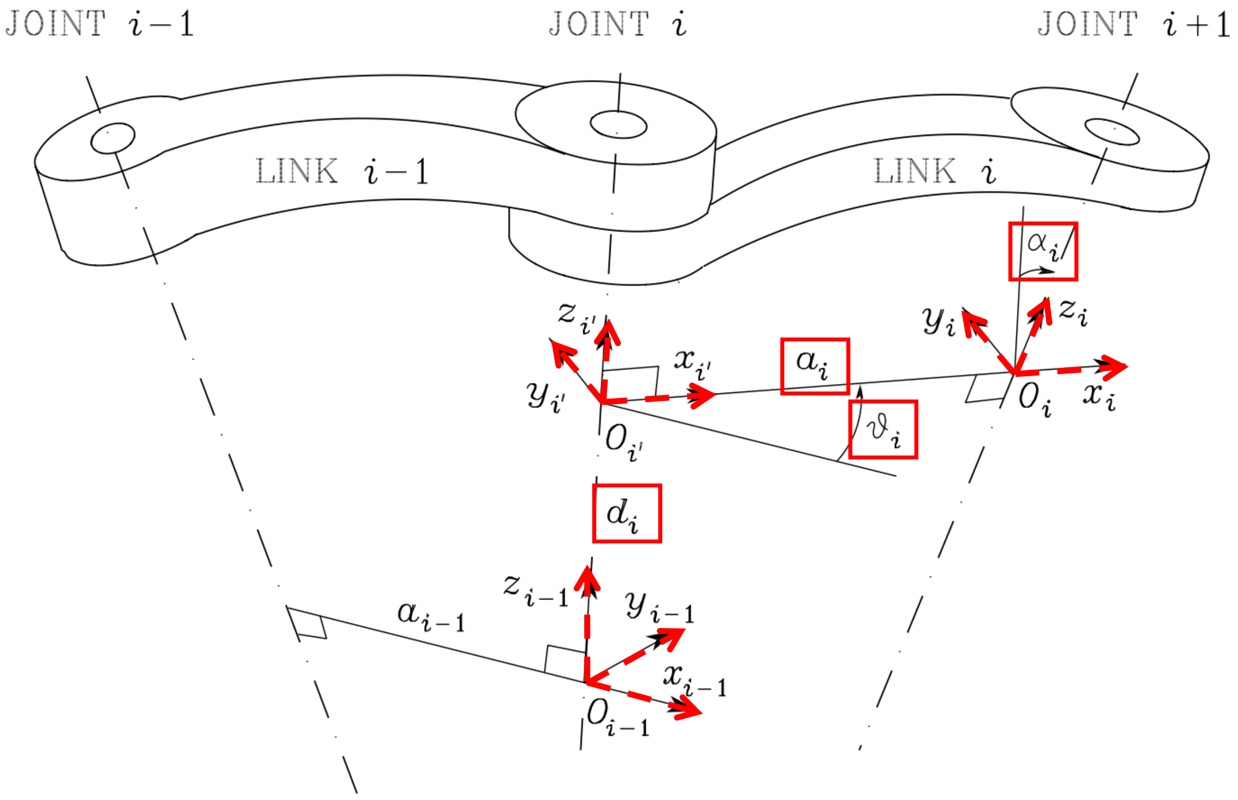
$$T_{i'}^{i-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the Spatial Transformations under Denavit–Hartenberg Convention?



$$R_{z,\vartheta_i} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are the Spatial Transformations under Denavit–Hartenberg Convention?



$$A_i^{i-1} = R_{z,\vartheta_i} T_{i'}^{i-1} T_i^{i'} R_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} c_{\alpha_i} & s_{\vartheta_i} s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i} c_{\alpha_i} & -c_{\vartheta_i} s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics: Calculate the Pose of a Robot End-Effector given the Configuration

- We known:

Transformation from joint i to frame $i-1$

$$\boxed{A_i^{i-1}} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c_{\alpha_i} & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation from joint n to the robot base

Transformation from the end-effector to the robot base

$$\boxed{A_e^b(\boldsymbol{q})} = \boxed{A_0^b} A_1^0(q_1) \ldots \boxed{A_n^{n-1}(q_n)}\boxed{A_e^n}$$

Configuration of joint n

Transformation from the end-effector to joint n

$$\boldsymbol{p}^b = A_0^b A_1^0(q_1) \ldots A_n^{n-1}(q_n) A_e^n \boldsymbol{p}^e$$

- Since the end-effector pose at the end-effector frame is constant, we can re-write the end-effector pose at the robot base frame as a function of $\boldsymbol{q}$: $f(\boldsymbol{q})$

Inverse Kinematics: Calculate the Robot Configuration given a Pose of the End-Effector

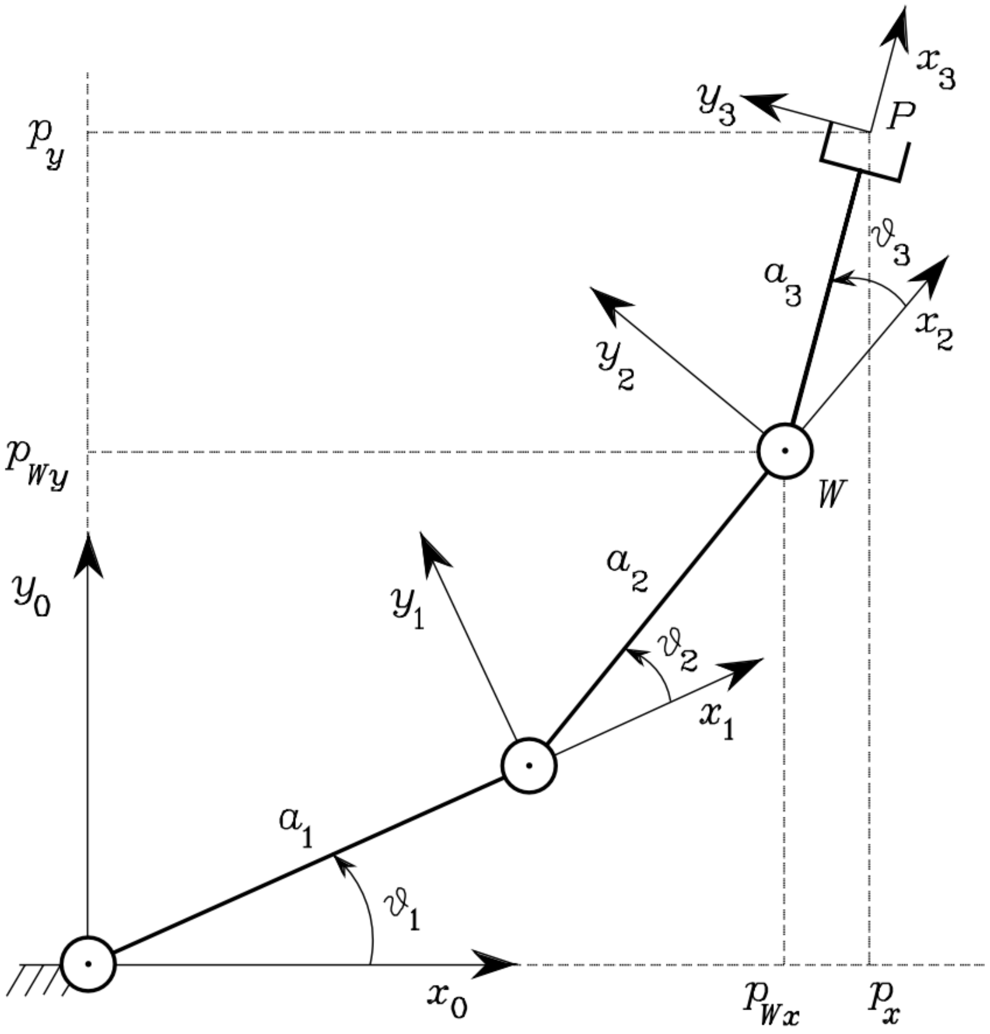
- Forward kinematic:

$$\textbf{end}{-}\textbf{effector pose} = f(\boldsymbol{q})$$

- Inverse kinematic:

$$\boldsymbol{q} = f^{-1}(\textbf{end}{-}\textbf{effector pose})$$

Analytical Solution of Inverse Kinematic: An Example of 3-link Planar Arm



Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

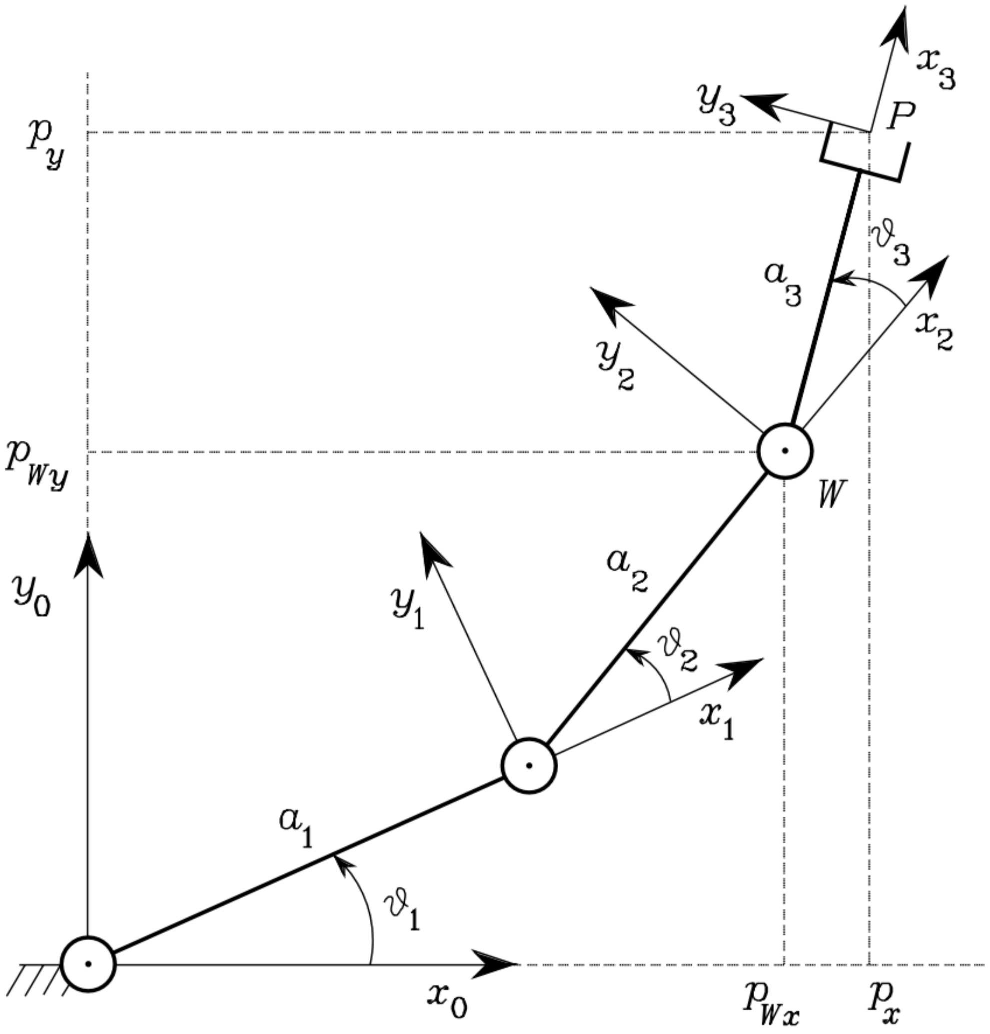
$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3.$$

$\cos(\vartheta_1 + \vartheta_2 + \vartheta_3)$

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 \boxed{c_{123}} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robotics: Modeling, Planning and Control. Siciliano et al.

Analytical Solution of Inverse Kinematic: An Example of 3-link Planar Arm



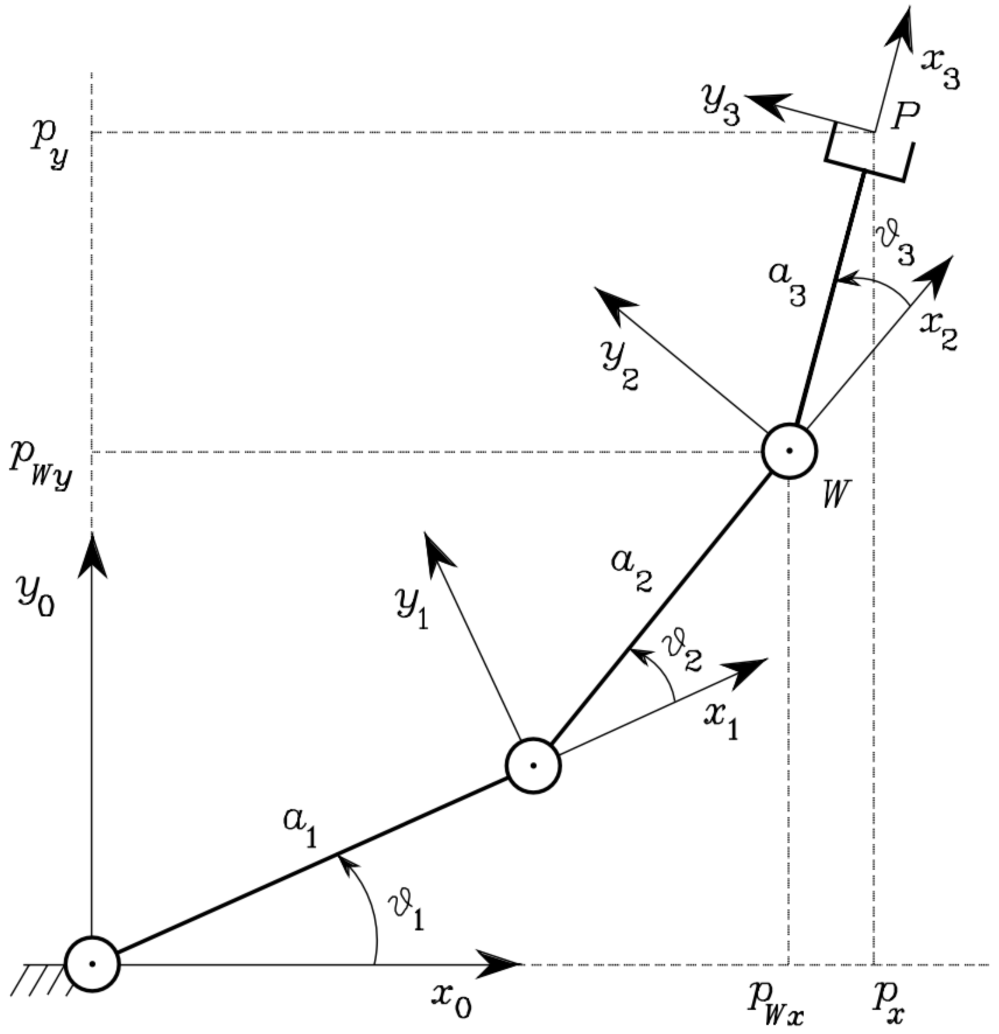
$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T}_e^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{end−effector pose} = T_3^0 T_e^3 \boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

Origin of the end-effector frame

Analytical Solution of Inverse Kinematic: An Example of 3-link Planar Arm



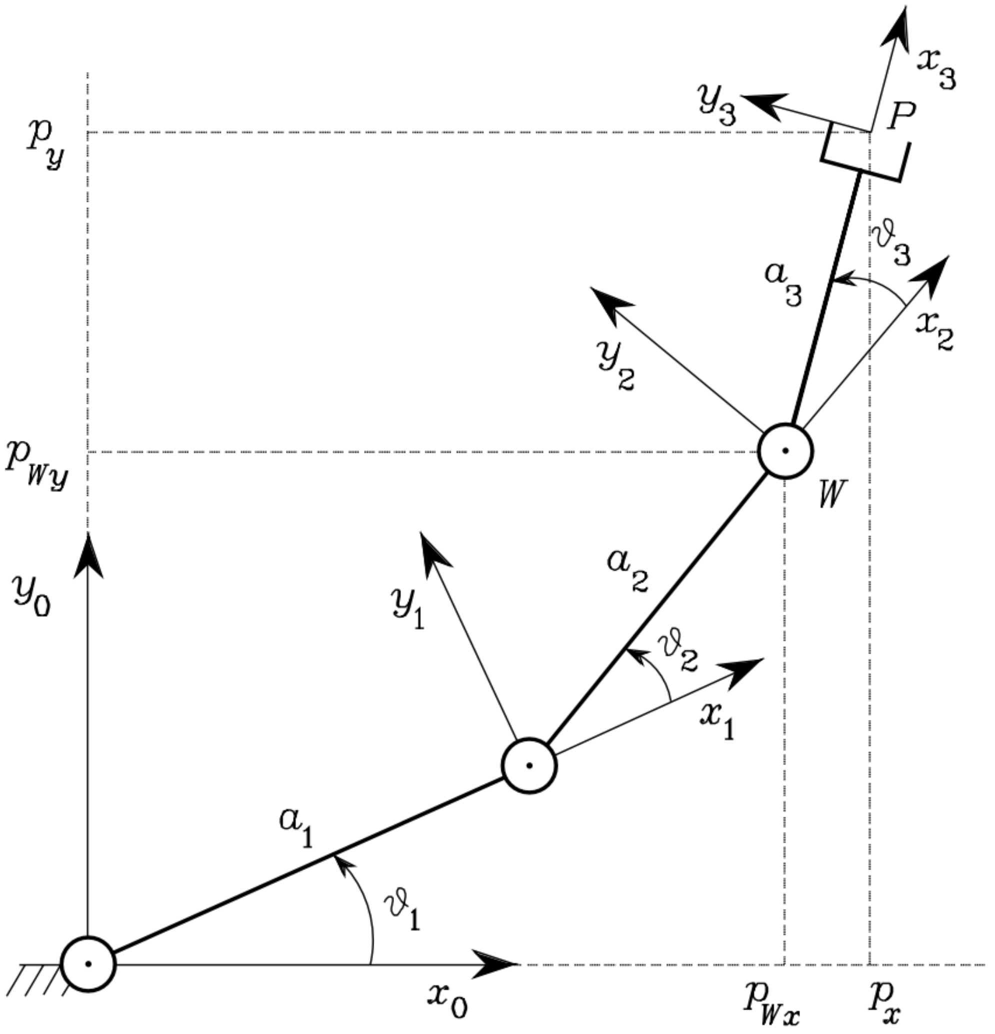
$$
\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0\boldsymbol{A}_2^1\boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\boldsymbol{T}_e^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\text{end}-\text{effector pose} = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix}
$$

$$
= \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}
$$

Analytical Solution of Inverse Kinematic: An Example of 3-link Planar Arm



$$\begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

1. $p_{W_x} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$
2. $p_{W_y} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12}$
3. From 1. and 2., ${p_{W_x}}^2 + {p_{W_y}}^2 = a_1^2 + a_2^2 + 2a_1 c_1 a_2 (c_1 c_2 - s_1 s_2) + 2a_1 s_1 a_2 (s_1 c_2 + c_1 s_2) = a_1^2 + a_2^2 + 2a_1 a_2 c_2$

We have $c_2 = \frac{{p_{W_x}}^2 + {p_{W_y}}^2 - a_1^2 + a_2^2}{2a_1 a_2}$

We have $s_2 = \pm\sqrt{1 - c_2^2}$ and $\vartheta_2 = Atan2\,(s_2, c_2)$

Robotics: Modeling, Planning and Control. Siciliano et al.

Numerical Solution of Inverse Kinematic

- Newton-Raphson Method:

$$x_d = f(\theta_d) = f(\theta^0) + \underbrace{\left.\frac{\partial f}{\partial \theta}\right|_{\theta^0}}_{J(\theta^0)} \underbrace{(\theta_d - \theta^0)}_{\Delta\theta} + \text{h.o.t.},$$

?

$$\Delta\theta = J^{-1}(\theta^0)\boxed{\left(x_d - f(\theta^0)\right)}.$$

This term include rotation and translation--
spatial transformation SE(3)

Numerical Solution of Inverse Kinematic

- Newton-Raphson Method:

$$x_d = f(\theta_d) = f(\theta^0) + \underbrace{\left.\frac{\partial f}{\partial \theta}\right|_{\theta^0}}_{J(\theta^0)} \underbrace{(\theta_d - \theta^0)}_{\Delta\theta} + \text{h.o.t.},$$

Solving J^{-1} using Denavit–Hartenberg Convention is overly complex…

$$\Delta\theta = \boxed{J^{-1}(\theta^0)}\,\boxed{\left(x_d - f(\theta^0)\right)}.$$

This term include rotation and translation-- spatial transformation SE(3)

What are Representations of Spatial Transformations?

- Desired properties:
  - ➢ Representing the forward kinematics of an open chain system as a product of transformations
  - ➢ Systemic derivation of the Jacobian that connects spatial transformations to joint angles
  - ➢ Unified representations of different types of robot joints
  - ➢ No need for solving inverse Jacobian matrix

We Can Associate One Representation of Spatial Transformation with Joint Angles in Linear Forms

- The twist coordinate (exponential of a twist is a rigid transformation matrix):

$$\xi^s = \underbrace{\xi_1 \dot{\theta}_1}_{\mathrm{J}_{\xi_1}(\theta)} + \underbrace{\mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2}_{\mathrm{J}_{\xi_2}(\theta)} \dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2}} \hat{\xi}_3}_{\mathrm{J}_{\xi_3}(\theta)} \dot{\theta}_3 + \cdots$$

$$= [\mathrm{J}_{\xi_1} \quad \mathrm{J}_{\xi_2} \quad \mathrm{J}_{\xi_3} \quad \ldots] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \vdots \end{bmatrix} = \mathbf{J}_{\xi} \dot{\boldsymbol{\theta}}$$

Solve Inverse Kinematic without Calculating the Inverse Jacobian Matrix

- Newton-Raphson Method:

$$\Delta\theta = J^{-1}(\theta^0)\left(x_d - f(\theta^0)\right).$$

$$\Downarrow$$

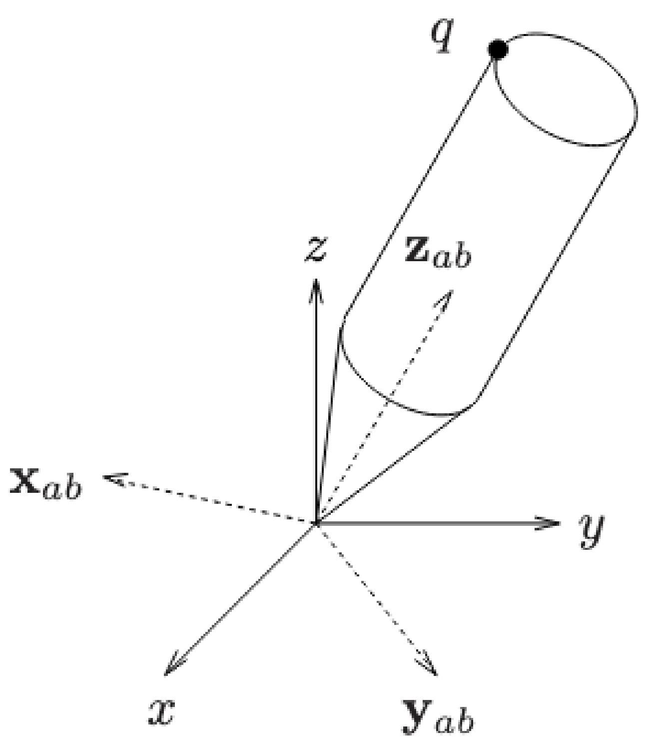
$$\Delta\theta = \boldsymbol{J}_{\xi}^{-1}\left(\xi^{\Delta}\right), \text{where } \xi^{\Delta} = \log\left(T_{current}^{-1}(\theta)T_{target}\right)$$

What is Twist?

Rotational Motion in $\mathbb{R}^3$

- Rotation matrix: $R_b^a = [\boldsymbol{x}_{ab}, \boldsymbol{y}_{ab}, \boldsymbol{z}_{ab}]$
- Properties of rotation matrices:
  - Orthonormality: $RR^\intercal = R^\intercal R = I$
  - $\det R = +1$

Why?



Rotational Motion in $\mathbb{R}^3$

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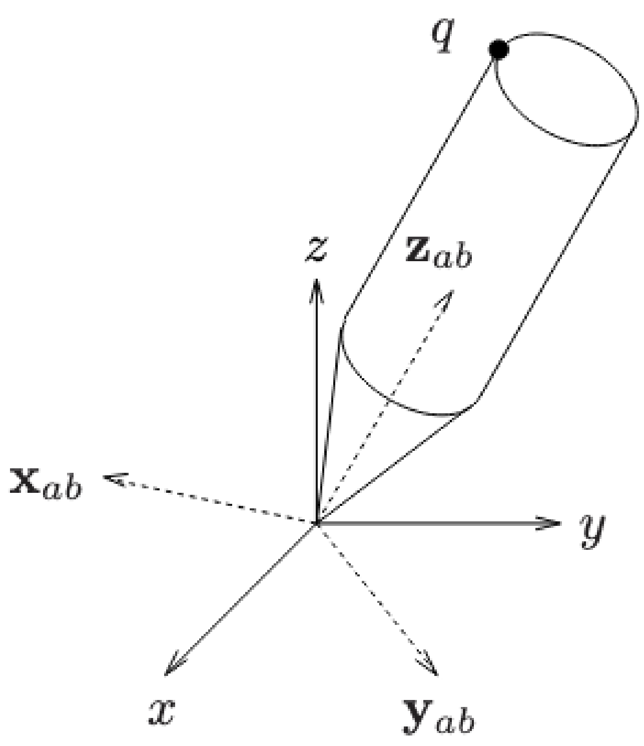
- Properties of rotation matrices:
  - Orthonormality: $RR^\intercal = R^\intercal R = I$
  - $\det R = +1$

$$\boxed{\det R = {r_1}^\intercal (r_2 \times r_3) = {r_1}^\intercal r_1 = 1}$$

- Rotation matrices $\mathbb{R}^{n\times n}$ form a special group:
  $SO(n) = \{\mathbb{R}^{n\times n}: RR^\intercal = I, \det R = +1\}$
- Composability: $R_1^3 = R_2^3 R_1^2$

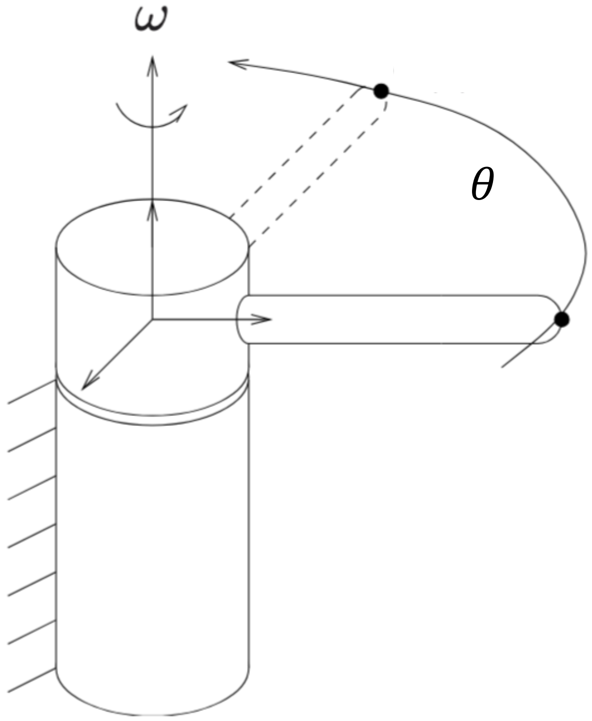


How to connect rotation matrices with robot joint angles?

Exponential Coordinates for Rotation

- A more geometrical description of "a rotation":
  - ➢ an axis of rotation $\boldsymbol{\omega}$ that specifies the direction of rotation
  - ➢ the angle of rotation $\boldsymbol{\theta}$

- Can we derive the rotation matrix from $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$?



A Mathematical Introduction to Robotic Manipulation. Murray, Li and Sastry.

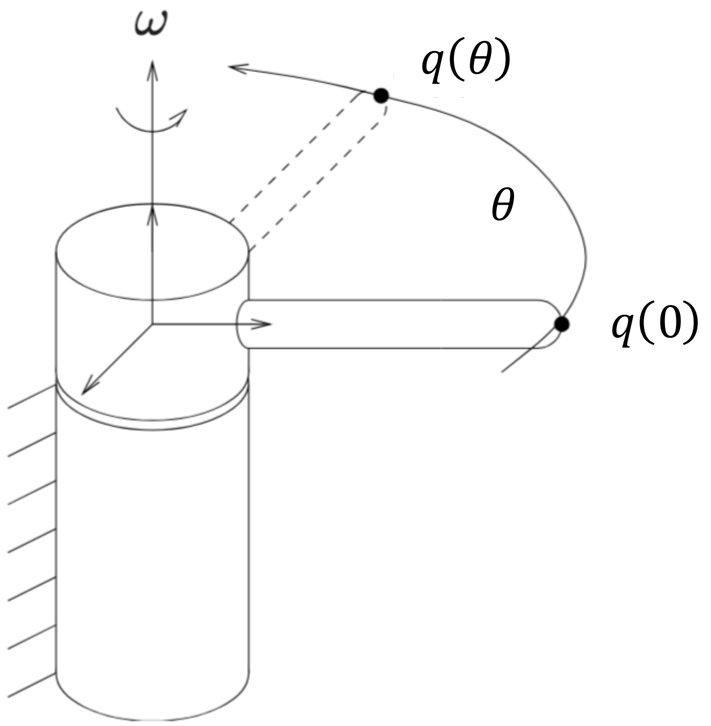
Exponential Coordinates for Rotation

- Prerequisite: cross products can be represented as matrix multiplications

$\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{a}}\boldsymbol{b}$, where skew matrix $\hat{\boldsymbol{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

- If we rotate the body at constant velocity about the axis $\boldsymbol{\omega}$, the velocity of the point:

$$\dot{\boldsymbol{q}}(\theta) = \boldsymbol{\omega} \times \boldsymbol{q}(\theta) = \hat{\boldsymbol{\omega}}\boldsymbol{q}(\theta)$$
$$\Rightarrow \boldsymbol{q}(\theta) = e^{\hat{\boldsymbol{\omega}}\theta}\boldsymbol{q}(0)$$



Exponential Coordinates for Rotation

- Properties of skew matrix $\hat{\boldsymbol{a}}$:
  - The set of 3×3 real skew-symmetric matrices composes $so(3)$
  - $\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{a}}\boldsymbol{b}$
  - $\boldsymbol{a}^{\mathsf{T}} = -\boldsymbol{a}$
- We can rewrite $e^{\hat{\boldsymbol{\omega}}\theta}$:

If $\|\boldsymbol{\omega}\| = 1$:
$\hat{\boldsymbol{\omega}}^2 = \boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}} - \|\boldsymbol{\omega}\|^2 I$
$\hat{\boldsymbol{\omega}}^3 = -\|\boldsymbol{\omega}\|^2 \hat{\boldsymbol{\omega}}$

$$e^{\hat{\boldsymbol{\omega}}\theta} = I + \theta\hat{\boldsymbol{\omega}} + \frac{\theta^2}{2!}\hat{\boldsymbol{\omega}}^2 + \frac{\theta^3}{3!}\hat{\boldsymbol{\omega}}^3 + \cdots$$

$$\Rightarrow e^{\hat{\boldsymbol{\omega}}\theta} = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\hat{\boldsymbol{\omega}} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)\hat{\boldsymbol{\omega}}^2$$

$$\Rightarrow e^{\hat{\boldsymbol{\omega}}\theta} = I + \sin\theta\,\hat{\boldsymbol{\omega}} + (1 - \cos\theta)\hat{\boldsymbol{\omega}}^2$$

Exponential Coordinates for Rotation

- Given $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$, the exponential $\boldsymbol{e^{\widehat{\omega}\theta}} \in SO(3)$
  - $\left[e^{\widehat{\boldsymbol{\omega}}\theta}\right]^{-1} = e^{-\widehat{\boldsymbol{\omega}}\theta} = e^{\widehat{\boldsymbol{\omega}}^{\intercal}\theta} = \left[e^{\widehat{\boldsymbol{\omega}}\theta}\right]^{\intercal}$
  - $\det e^{\widehat{\boldsymbol{\omega}}\theta} = +1$ (based on the continuity of exponential map and the fact that $\det e^{0} = 1$)
- Given a rotation matrix R, there exists $\boldsymbol{\omega} \in \mathbb{R}^3$ and $\boldsymbol{\theta}$, such that $R = e^{\widehat{\boldsymbol{\omega}}\boldsymbol{\theta}}$

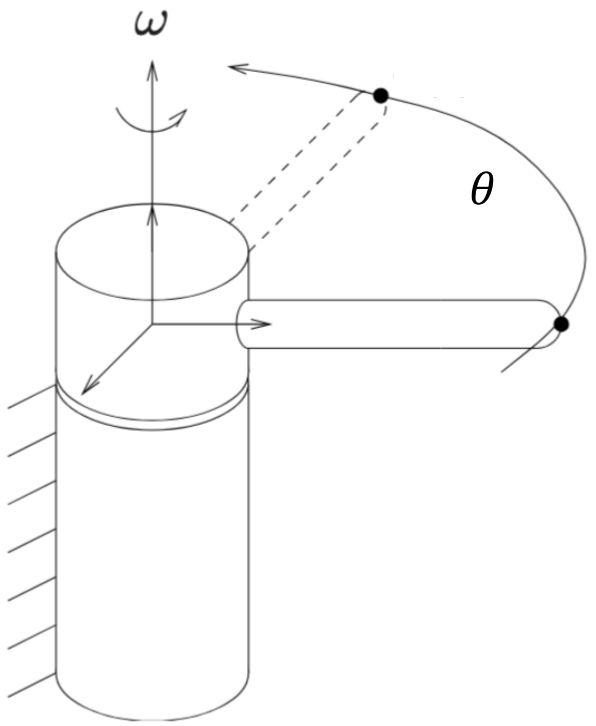
$$e^{\widehat{\boldsymbol{\omega}}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix} \longleftrightarrow R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

all we need is to find the solution

$$c_\theta = \cos\theta\,,\ s_\theta = \sin\theta\,,\ v_\theta = 1 - \cos\theta$$

Quick summary

- Exponentials provide a more geometrical description of “a rotation”:
  - ➢ an axis of rotation $\boldsymbol{\omega}$ that specifies the direction of rotation
  - ➢ the angle of rotation θ
  - ➢ $e^{\widehat{\boldsymbol{\omega}}\boldsymbol{\theta}} \in SO(3)$
  - ➢ Given a rotation matrix R, there exists $\boldsymbol{\omega} \in \mathbb{R}^3$ and θ, such that $R = e^{\widehat{\boldsymbol{\omega}}\boldsymbol{\theta}}$
- We can easily represent the revolute joint with such representations



Rigid Motion in $\mathbb{R}^3$

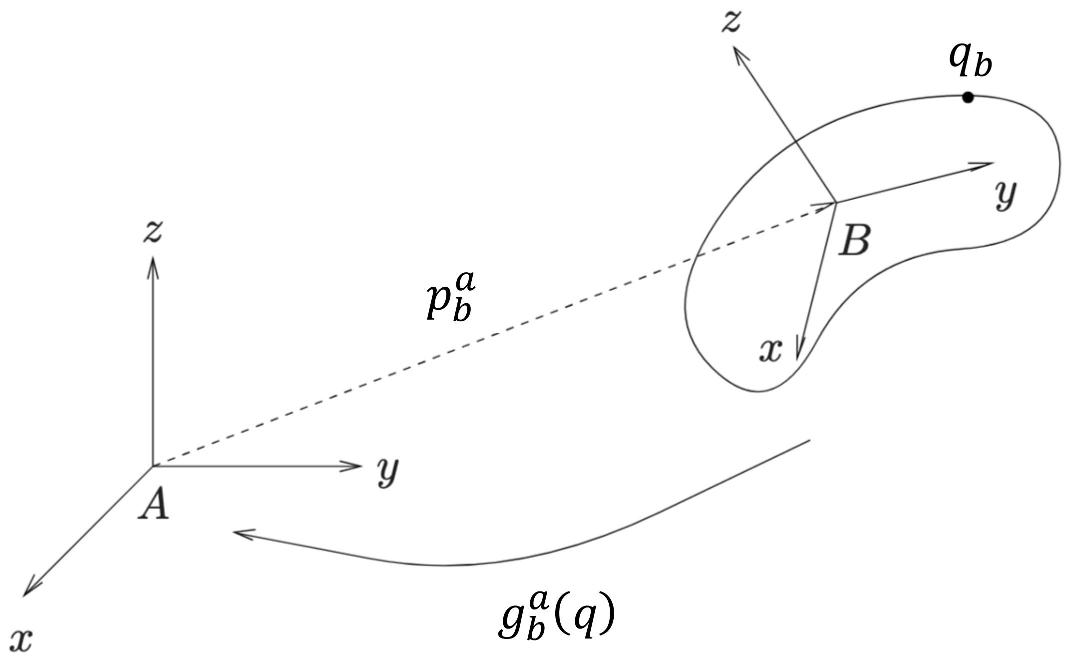
- Rotation matrix: $R_b^a = [\boldsymbol{x}_{ab}, \boldsymbol{y}_{ab}, \boldsymbol{z}_{ab}]$
- Translation: p_b^a

- Rigid transformation from frame b to a:

$$g_b^a(q) = q_a = p_b^a + R_b^a q_b$$

or in homogeneous representations

$$\underbrace{\begin{bmatrix} q_a \\ 1 \end{bmatrix}}_{\overline{q_a}} = \underbrace{\begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}}_{\bar{g}_b^a} \underbrace{\begin{bmatrix} q_b \\ 1 \end{bmatrix}}_{\overline{q_b}}$$



Rigid Motion in $\mathbb{R}^3$

- Rigid transformation from frame b to a:

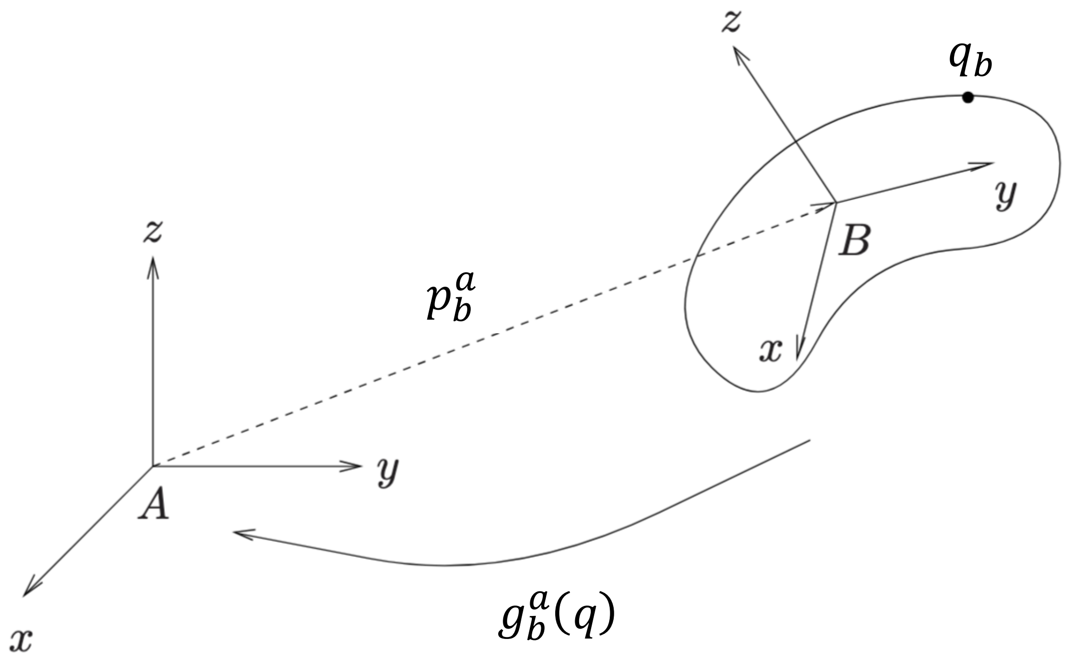
$$\underbrace{\begin{bmatrix} q_a \\ 1 \end{bmatrix}}_{\overline{q_a}} = \underbrace{\begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}}_{\bar{g}_b^a} \underbrace{\begin{bmatrix} q_b \\ 1 \end{bmatrix}}_{\overline{q_b}}$$

- Rigid transformations form a special group:

$$SE(n) = \{(p, R): p \in \mathbb{R}^n, R \in SO(n)\}$$

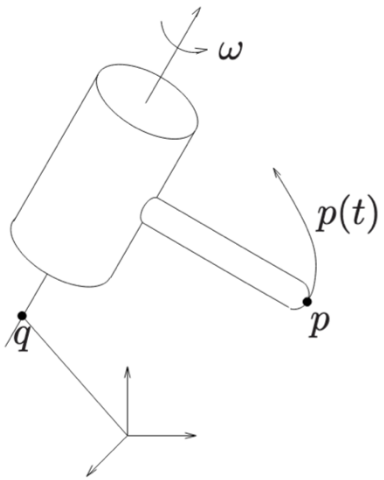
- Composability: $\bar{g}_1^3 = \bar{g}_2^3 \bar{g}_1^2$
- The inverse of $g \in SE(3)$ still belongs to $SE(3)$:

$$g^{-1} = \begin{bmatrix} R^\intercal & -R^\intercal p \\ 0 & 1 \end{bmatrix}$$



How to connect rigid transformations with robot configurations?

More Geometrical Representations for Rigid Transformation



- Pure rotation

$$\dot{\boldsymbol{p}}(\theta) = \boldsymbol{\omega} \times (\boldsymbol{p}(\theta) - \boldsymbol{q})$$

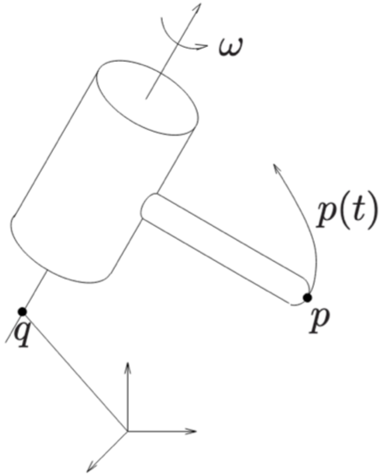
$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{\boldsymbol{\omega}} & -\boldsymbol{\omega} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$

$$\hat{\xi} = \begin{bmatrix} \widehat{\boldsymbol{\omega}} & -\boldsymbol{\omega} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \dot{\overline{\boldsymbol{p}}} = \hat{\xi}\overline{\boldsymbol{p}} \Rightarrow \overline{\boldsymbol{p}}(\theta) = \boldsymbol{e}^{\hat{\xi}\theta}\overline{\boldsymbol{p}}(\boldsymbol{0})$$

A Mathematical Introduction to Robotic Manipulation. Murray, Li and Sastry.

Exponential Coordinates for Rigid Transformation



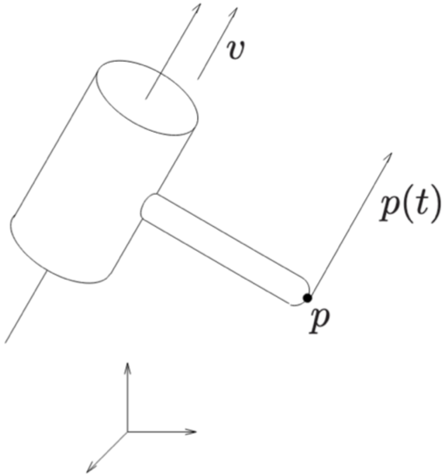
- Pure rotation

$$\dot{\boldsymbol{p}}(\theta) = \boldsymbol{\omega} \times (\boldsymbol{p}(\theta) - \boldsymbol{q})$$

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{\boldsymbol{\omega}} & -\boldsymbol{\omega} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$

$$\hat{\xi} = \begin{bmatrix} \widehat{\boldsymbol{\omega}} & -\boldsymbol{\omega} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \dot{\overline{\boldsymbol{p}}} = \hat{\xi} \overline{\boldsymbol{p}} \Rightarrow \overline{\boldsymbol{p}}(\theta) = \boldsymbol{e}^{\hat{\xi}\theta} \overline{\boldsymbol{p}}(\boldsymbol{0})$$



- Pure translation

$$\dot{\boldsymbol{p}}(\theta) = \boldsymbol{v}$$

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$

$$\hat{\xi} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \dot{\overline{\boldsymbol{p}}} = \hat{\xi} \overline{\boldsymbol{p}} \Rightarrow \overline{\boldsymbol{p}}(\theta) = \boldsymbol{e}^{\hat{\xi}\theta} \overline{\boldsymbol{p}}(\boldsymbol{0})$$

A Mathematical Introduction to Robotic Manipulation. Murray, Li and Sastry.

The $\hat{\xi}$ Matrix and the Twist

- $\hat{\xi}$ matrices form a special group

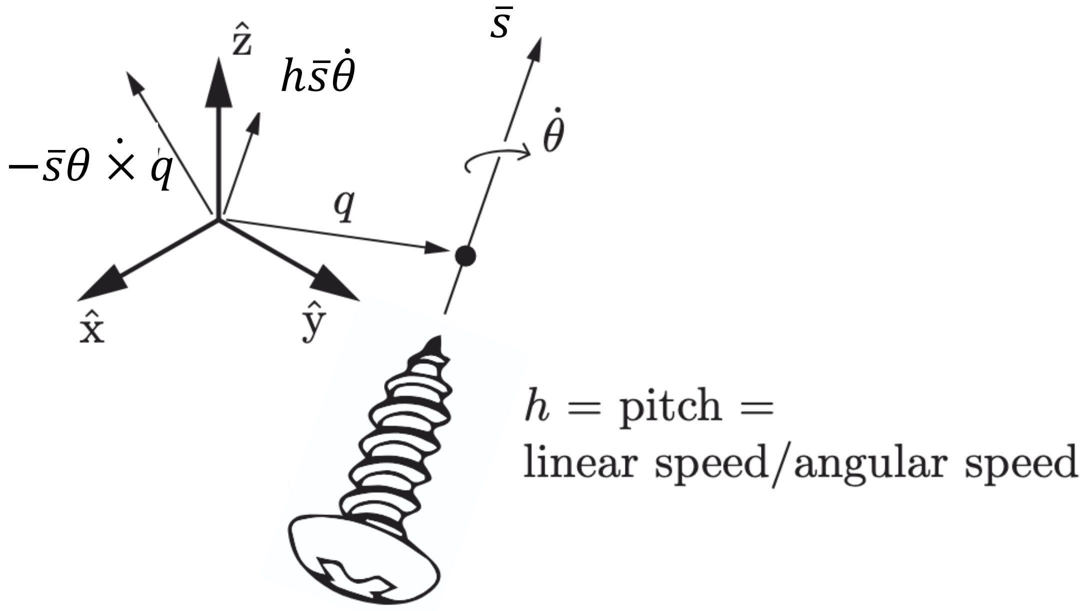
$$se(3) := \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} : v \in \mathbb{R}^3, \hat{\omega} \in so(3) \right\}$$

- We define $\xi := (v, \hat{\omega})$ as the twist coordinates of $\hat{\xi}$
- Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

We'll prove later!

The Screw Representation of Rigid Motion

- We can interpret a twist as a screw axis $\mathcal{S}$ and an angular velocity $\dot{\theta}$
- A screw axis $\mathcal{S} = \{q, \bar{s}, h\}$, where
  - ➢ q is any point on the axis
  - ➢ $\bar{s}$ is a unit vector in the direction of the axis
  - ➢ h is the "screw pitch", denoting the ratio of the linear velocity to the angular velocity $\dot{\theta}$

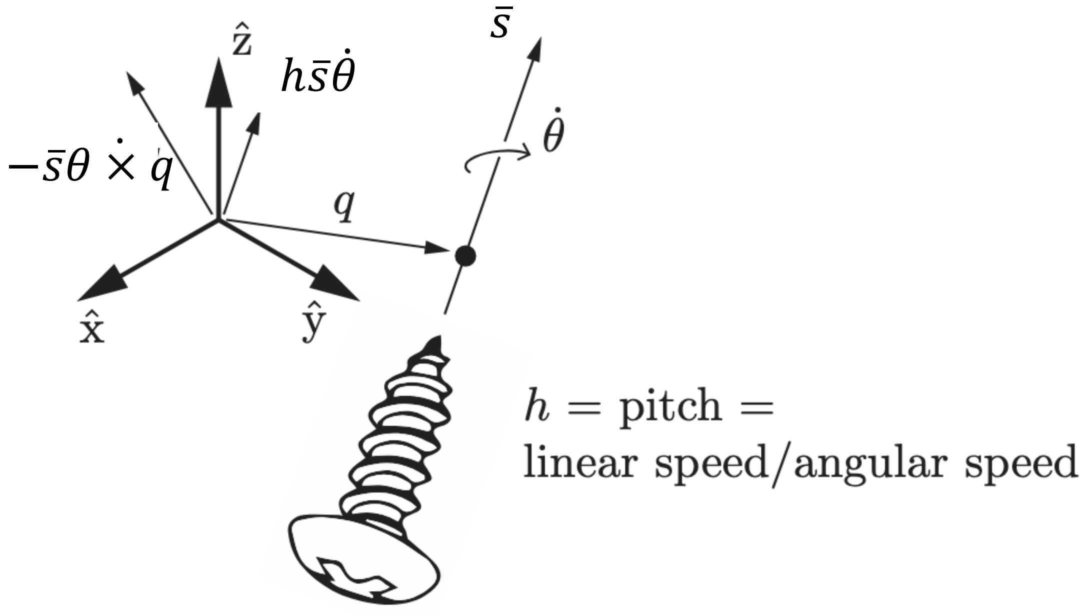


- $h = 0$, the screw motion denotes pure rotation (e.g. revolute joint)
- $h \to \infty$, the screw motion denotes pure translation (e.g. prismatic joint)

The Screw Representation of Rigid Motion

→ With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we define a twist:

$$\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta} \end{bmatrix}$$



The Screw Representation of Rigid Motion

→ With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we define a twist:

$$\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta} \end{bmatrix}$$

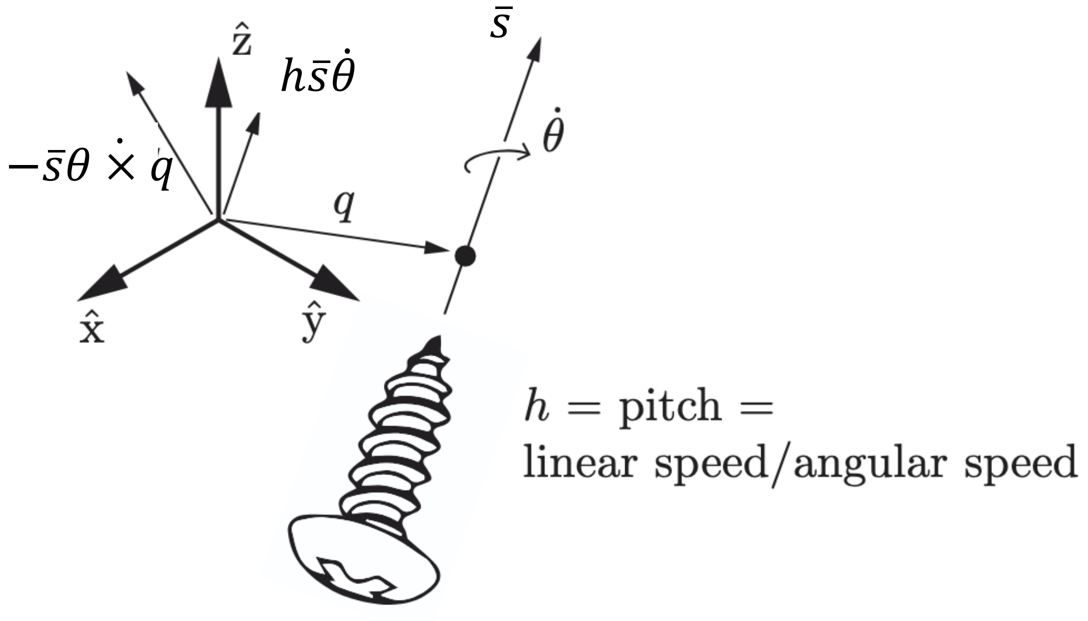
← For any twist ξ where $\omega \neq \mathbf{0}$, there exists an equivalent screw axis $\mathcal{S} = \{q, \bar{s}, h\}$ and velocity $\dot{\theta}$:

$$\bar{s} = \omega / \|\omega\|$$

$$\dot{\theta} = \|\omega\|$$

$$h = \hat{\omega}^\intercal v / \dot{\theta}$$

q is chosen accordingly



What Do We Need to Show?

- Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$
- With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we have a twist: $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta} \end{bmatrix}$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- If $\omega = \mathbf{0}$:

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & v \\ 0 & 0 \end{bmatrix} \Rightarrow \hat{\xi}^2 = \hat{\xi}^3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 0 & 0 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

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$$e^{\hat{\xi}\theta} = I + \theta\hat{\boldsymbol{\xi}} + \frac{\theta^2}{2!}\hat{\boldsymbol{\xi}}^2 + \frac{\theta^3}{3!}\hat{\boldsymbol{\xi}}^3 + \cdots = I + \theta\hat{\boldsymbol{\xi}} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{v}\theta \\ 0 & 0 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- If $\omega \neq \mathbf{0}$:

  First, we assume $\|\widehat{\omega}\| = 1$ and we can scale θ appropriately

  Next, we define a rigid transformation $g = \begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix}$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

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  First, we assume $\|\widehat{\omega}\| = 1$ and we can scale θ appropriately

  Next, we define a rigid transformation $g = \begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix}$

  Let $\widehat{\xi'} = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -\widehat{\omega}v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix}$

  $$= \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \widehat{\omega} & \widehat{\omega}\widehat{\omega}v + v \\ 0 & 0 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- If $\omega \neq \mathbf{0}$:

  First, we assume $\|\hat{\omega}\| = 1$ and we can scale θ appropriately

  Next, we define a rigid transformation $g = \begin{bmatrix} I & \hat{\omega}v \\ 0 & 1 \end{bmatrix}$

  Let $\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -\hat{\omega}v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \hat{\omega}v \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \hat{\omega}v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \hat{\omega}\hat{\omega}v + v \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{gathered} a \times (b \times c) \\ = b(a \cdot c) - c(a \cdot b) \end{gathered}}$$

$$= \begin{bmatrix} \hat{\omega} & \omega\omega^\top v - v\cancel{\omega^\top}\omega + \cancel{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \omega\omega^\top v \\ 0 & 0 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- If $\omega \neq \mathbf{0}$:

  First, we assume $\|\hat{\omega}\| = 1$ and we can scale θ appropriately

  Next, we define a rigid transformation $g = \begin{bmatrix} I & \hat{\omega}v \\ 0 & 1 \end{bmatrix}$

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$$= \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \hat{\omega}v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \hat{\omega}\hat{\omega}v + v \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} a \times (b \times c) \\ = b(a \cdot c) - c(a \cdot b) \end{array}$$

$$= \begin{bmatrix} \hat{\omega} & \omega\omega^\top v - \cancel{v\omega^\top\omega} + \cancel{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \omega\overset{h}{\boxed{\omega^\top v}} \\ 0 & 0 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- If $\omega \neq \mathbf{0}$:

  First, we assume $\|\widehat{\omega}\| = 1$ and we can scale θ appropriately

  Next, we define a rigid transformation $g = \begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix}$

  Let $\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -\widehat{\omega}v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & \widehat{\omega}v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \widehat{\omega} & \widehat{\omega}\widehat{\omega}v + v \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{\omega} & \omega\omega^\top v - v\cancel{\omega^\top}\omega + \cancel{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\omega} & \omega\overset{\hbar}{\boxed{\omega^\top v}} \\ 0 & 0 \end{bmatrix}$$

$a \times (b \times c)$
$= b(a \cdot c) - c(a \cdot b)$

What's next? We are going to derive $e^{\hat{\xi}\theta}$ from $e^{g\hat{\xi}'\theta g^{-1}}$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- Before continuing the proof, we first show that for any invertible matrix $g \in \mathbb{R}^{n\times n}$ and a matrix $\Lambda \in \mathbb{R}^{n\times n}$, we have $e^{g\Lambda g^{-1}} = g e^{\Lambda} g^{-1}$

$$\because e^{\Lambda} = I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots$$

$$\therefore e^{g\Lambda g^{-1}} = I + g\Lambda g^{-1} + \frac{g\Lambda g^{-1} g\Lambda g^{-1}}{2!} + \frac{g\Lambda g^{-1} g\Lambda g^{-1} g\Lambda g^{-1}}{3!} + \cdots$$

$$= I + g\Lambda g^{-1} + \frac{g\Lambda^2 g^{-1}}{2!} + \frac{g\Lambda^3 g^{-1}}{3!} + \cdots$$

$$= gg^{-1} + g\Lambda g^{-1} + \frac{g\Lambda^2 g^{-1}}{2!} + \frac{g\Lambda^3 g^{-1}}{3!} + \cdots = g e^{\Lambda} g^{-1}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- We have $e^{\hat{\xi}\theta} = e^{g\hat{\xi}'\theta g^{-1}} = ge^{\hat{\xi}'\theta}g^{-1}$
- What's $e^{\hat{\xi}'\theta}$?

  First, we know $\widehat{\omega}\omega = \omega \times \omega = 0$

  Second, we know $\hat{\xi}'^2 = \begin{bmatrix} \widehat{\omega} & \omega\hbar \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \widehat{\omega} & \omega\hbar \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\omega}^2 & \widehat{\omega}\omega\hbar \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}$

  $$\hat{\xi}'^3 = \begin{bmatrix} \widehat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix} \ldots$$

  $$e^{\hat{\xi}'\theta} = I + \theta\hat{\xi}' + \frac{\theta^2}{2!}\hat{\xi}'^2 + \frac{\theta^3}{3!}\hat{\xi}'^3 + \cdots = \begin{bmatrix} I + \frac{\widehat{\omega}\theta^2}{2!} + \frac{\widehat{\omega}\theta^3}{3!} + \cdots & \omega\hbar\theta + 0 + \cdots \\ 0 & 1 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- We have $e^{\hat{\xi}\theta} = e^{g\widehat{\xi'}\theta g^{-1}} = g e^{\widehat{\xi'}\theta} g^{-1}$
- What's $e^{\hat{\xi}\theta}$?

$$e^{\widehat{\xi'}\theta} = \begin{bmatrix} I + \frac{\widehat{\omega}^2}{2!} + \frac{\widehat{\omega}^3}{3!} + \cdots & \omega \hbar \theta + 0 + \cdots \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\widehat{\omega}\theta} & \omega \hbar \theta \\ 0 & 1 \end{bmatrix}$$

Let's put everything together

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\widehat{\omega}\theta} & \left(I - e^{\widehat{\omega}\theta}\right)\widehat{\omega}v + \omega\omega^{\intercal} v\theta \\ 0 & 1 \end{bmatrix}$$

Proof: Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$

- We have $e^{\hat{\xi}\theta} = e^{g\hat{\xi}'\theta g^{-1}} = g e^{\hat{\xi}'\theta} g^{-1}$
- What's $e^{\hat{\xi}\theta}$?

$$e^{\hat{\xi}'\theta} = \begin{bmatrix} I + \frac{\hat{\omega}^2}{2!} + \frac{\hat{\omega}^3}{3!} + \cdots & \omega\hbar\theta + 0 + \cdots \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & \omega\hbar\theta \\ 0 & 1 \end{bmatrix}$$

Let's put everything together

$$e^{\hat{\xi}\theta} = \begin{bmatrix} \boxed{e^{\hat{\omega}\theta}} & \left(I - e^{\hat{\omega}\theta}\right)\hat{\omega}v + \omega\omega^{\intercal}v\theta \\ 0 & 1 \end{bmatrix}$$

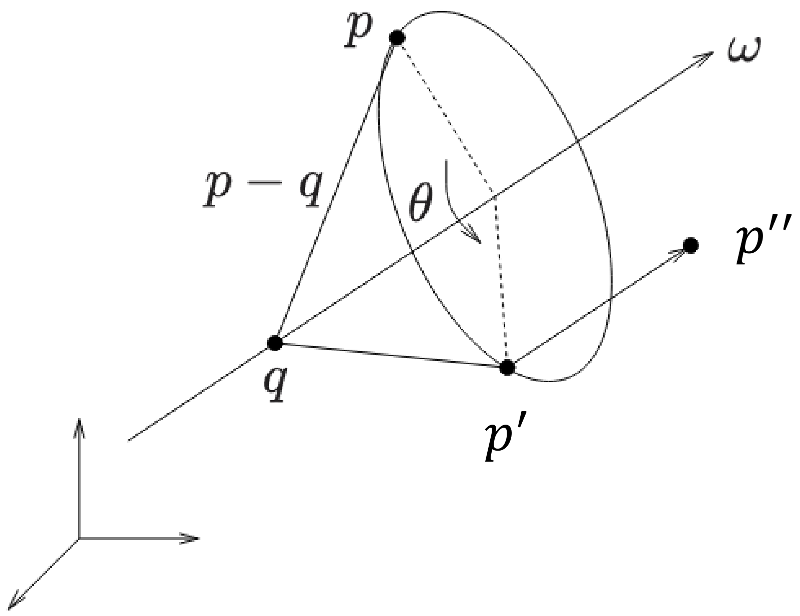
$SO(3)$

What Do We Need to Show?

- Given $\hat{\xi} \in se(3)$ and $\theta \in \mathbb{R}$, the exponential $e^{\hat{\xi}\theta} \in SE(3)$
- With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we have a twist: $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta} \end{bmatrix}$

Proof: With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we have a twist: $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta} \end{bmatrix}$

- Let's write down the rigid motion from point $\boldsymbol{p}$ to $\mathrm{g}(p) = p''$:



Proof: With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we have a twist: $\xi = \begin{bmatrix}\omega \\ v\end{bmatrix} = \begin{bmatrix}\bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta}\end{bmatrix}$

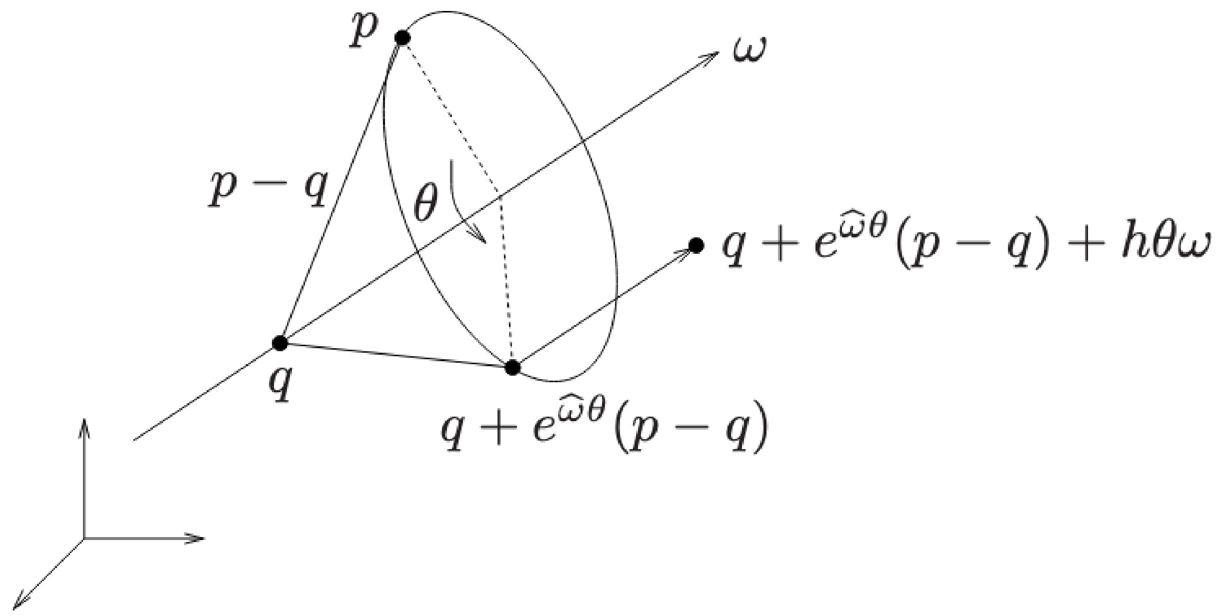
- Let's write down the rigid motion from point p to $\mathrm{g}(p) = p''$:

$$p' = q + e^{\widehat{\omega}\theta}(p - q)$$

$$p'' = p' + \omega h\theta$$

$$\mathrm{g}(p) = q + e^{\widehat{\omega}\theta}(p - q) + \omega h\theta$$

$$\Rightarrow \mathrm{g}\begin{bmatrix}p \\ 1\end{bmatrix} = \begin{bmatrix}e^{\widehat{\omega}\theta} & \left(I - e^{\widehat{\omega}\theta}\right)q + \omega h\theta \\ 0 & 1\end{bmatrix}\begin{bmatrix}p \\ 1\end{bmatrix}$$



Proof: With $\mathcal{S} = \{q, \bar{s}, h\}$ and $\dot{\theta}$, we have a twist: $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \bar{s}\dot{\theta} \\ -\bar{s}\dot{\theta} \times q + h\bar{s}\dot{\theta} \end{bmatrix}$

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$$\Rightarrow \mathrm{g}\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\widehat{\omega}\theta} & \left(I - e^{\widehat{\omega}\theta}\right)q + \omega h\theta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

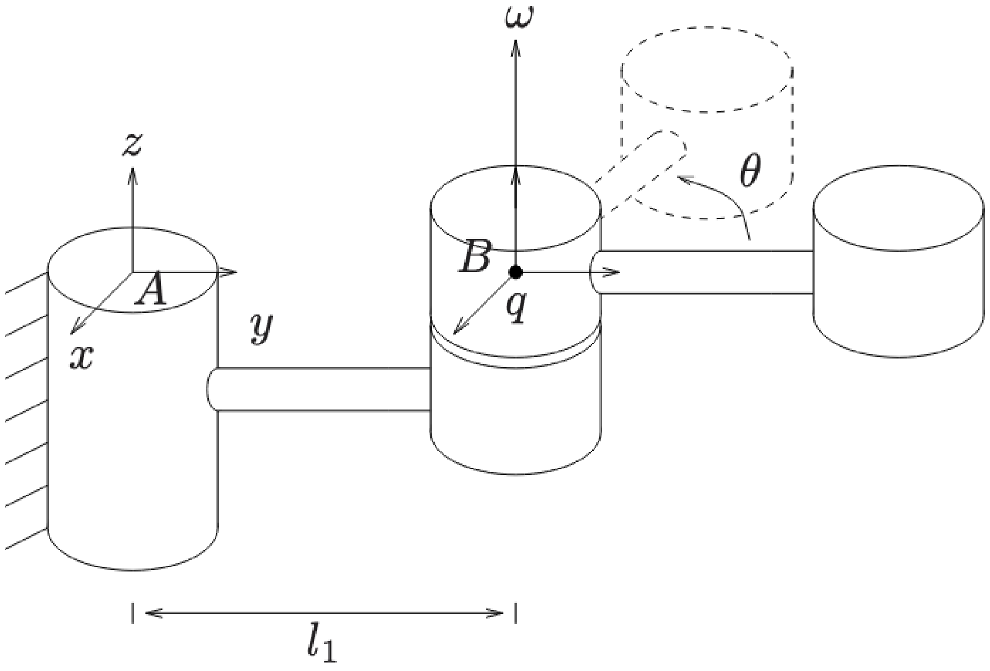
- We know:

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\widehat{\omega}\theta} & \left(I - e^{\widehat{\omega}\theta}\right)\widehat{\omega}v + \omega\omega^{\top}v\theta \\ 0 & 1 \end{bmatrix}$$

Equivalent if $v = -\omega \times q + \omega h$

Example: Revolute Joint

- The joint only rotates $(h = 0)$ at $q = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$
- Rotation axis: $\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- We have $v = -\omega \times q + \omega h = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$
- We have twist $\boldsymbol{\xi} = \begin{bmatrix} \omega \\ v \end{bmatrix}$

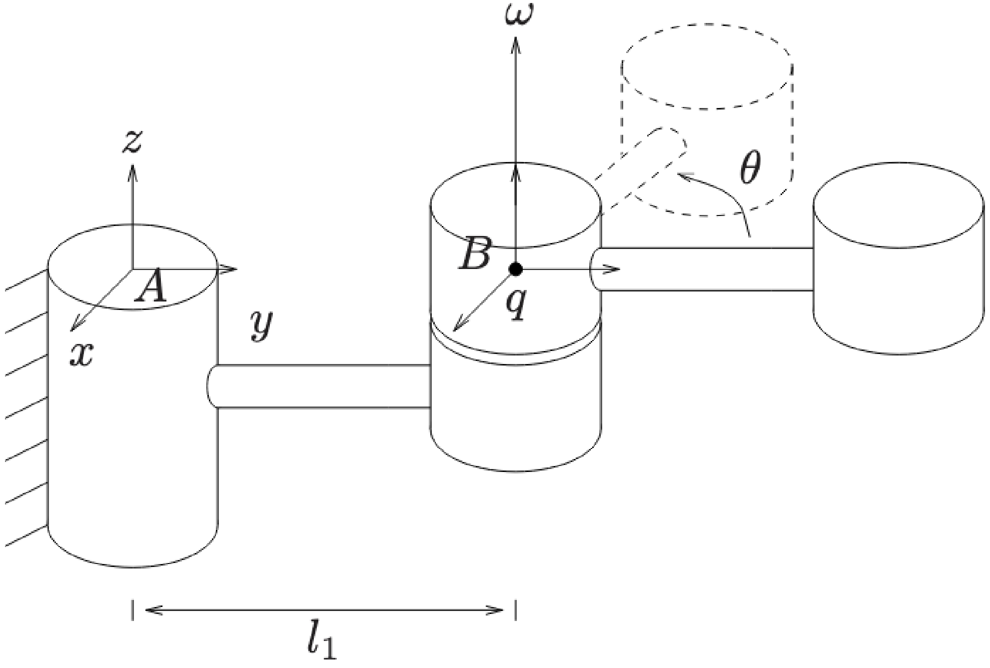


Example: Revolute Joint

- The exponential:

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\widehat{\omega}\theta} & \left(I - e^{\widehat{\omega}\theta}\right)\widehat{\omega}v + \omega\omega^{\intercal}v\theta \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & l_1 \sin\theta \\ \sin\theta & \cos\theta & 0 & l_1(1-\cos\theta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Previously, We Discussed Inverse Kinematic

- Connecting a twist and joint angles in linear forms:

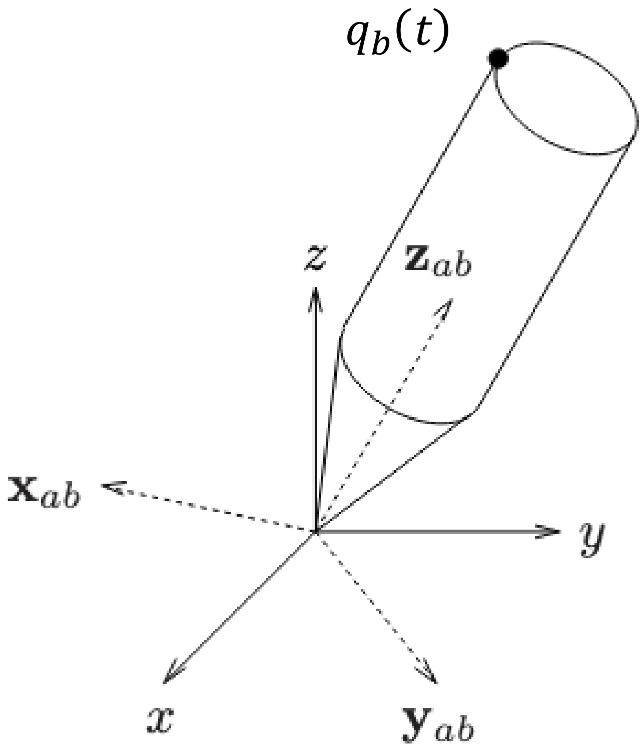
$$\xi_S = \underbrace{\mathcal{S}_1}_{J_{s1}} \dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}} \dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}} \dot{\theta}_3 + \cdots$$

$$\begin{aligned} \xi_S &= \begin{bmatrix} J_{s1} & J_{s2}(\theta) & \cdots & J_{sn}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \\ &= J_s(\theta)\dot{\theta}. \end{aligned}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rotational velocity: we know $q_a(t) = R_b^a q_b(t)$

$$\frac{dq_a(t)}{dt} = \underbrace{\frac{dR_b^a}{dt}}_{\dot{R}_b^a} q_b(t) + R_b^a \cancel{\frac{dq_b(t)}{dt}}$$



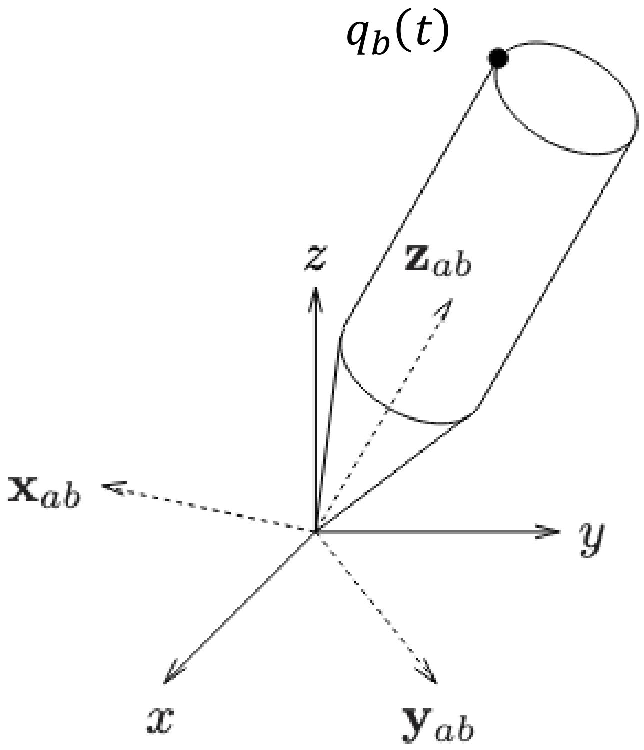
Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rotational velocity: we know $q_a(t) = R_b^a q_b(t)$

$$\frac{dq_a(t)}{dt} = \dot{R}_b^a q_b(t)$$

1. $$\frac{dq_a(t)}{dt} = \dot{R}_b^a q_b(t) = \dot{R}_b^a {R_b^a}^{-1} R_b^a q_b(t)$$
$$= \underbrace{\dot{R}_b^a {R_b^a}^{-1}}_{} q_a(t)$$

$\widehat{\omega}_{ab}^s$: instantaneous angular velocity of the object seen from Spatial (A) frame



Association of Velocities of a Rigid Body at Different Coordinate Frames

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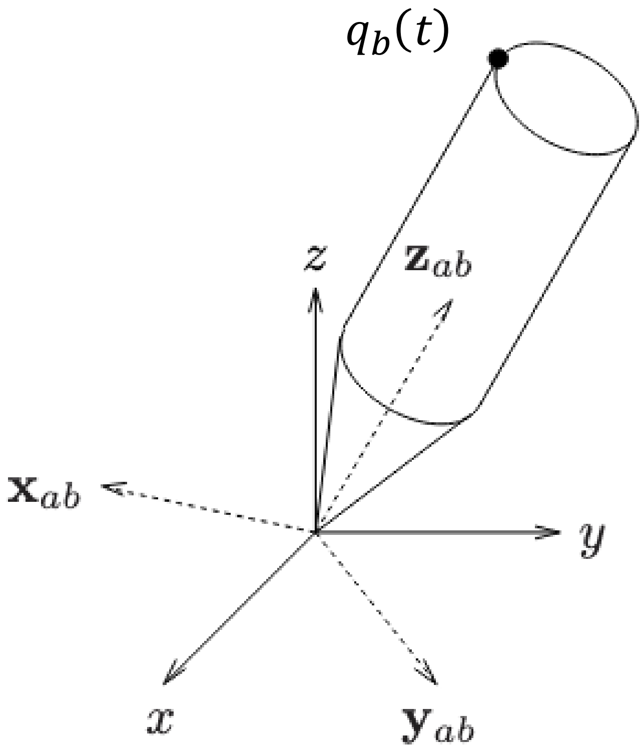
1. $\frac{dq_a(t)}{dt} = \dot{R}_b^a {R_b^a}^{-1} q_a(t) = \widehat{\omega}_{ab}^s q_a(t)$

2. $\frac{dq_a(t)}{dt} = \dot{R}_b^a q_b(t) = R_b^a \underbrace{{R_b^a}^{-1} \dot{R}_b^a}_{} q_b(t)$

$\widehat{\omega}_{ab}^b$: instantaneous angular velocity of the object seen from instantaneous body frame coincided with B frame at time t



Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rotational velocity: we know $q_a(t) = R_b^a q_b(t)$

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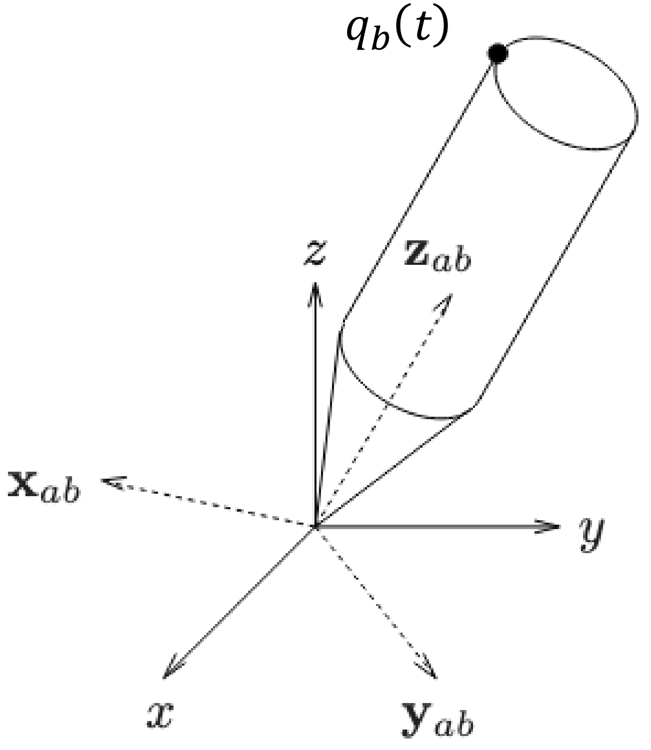
1. $\frac{dq_a(t)}{dt} = \dot{R}_b^a {R_b^a}^{-1} q_a(t) = \widehat{\omega}_{ab}^s q_a(t)$
2. $\frac{dq_a(t)}{dt} = R_b^a {R_b^a}^{-1} \dot{R}_b^a q_b(t) = R_b^a \widehat{\omega}_{ab}^b q_b(t)$

- We can associate angular velocity observed at the spatial and body frame:

$$\widehat{\omega}_{ab}^b = {R_b^a}^{-1} \widehat{\omega}_{ab}^s R_b^a$$



Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rigid body velocity:

  rigid motion: $g_b^a = \begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}$

  we know $\begin{bmatrix} q_a(t) \\ 1 \end{bmatrix} = g_b^a \begin{bmatrix} q_b(t) \\ 1 \end{bmatrix} \Rightarrow \tilde{q}_a(t) = g_b^a \tilde{q}_b(t)$

  we obtain twist observed at the spatial frame:

$$\frac{d\tilde{q}_a(t)}{dt} = \dot{g}_b^a \tilde{q}_b(t)$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

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  we obtain twist observed at the spatial frame:

$$\frac{d\tilde{q}_a(t)}{dt} = \dot{g}_b^a \tilde{q}_b(t) = \dot{g}_b^a {g_b^a}^{-1} g_b^a \tilde{q}_b(t) = \underbrace{\dot{g}_b^a {g_b^a}^{-1}}_{} \tilde{q}_a(t)$$

$\hat{\xi}_{ab}^s$: instantaneous twist of the object seen from spatial (A) frame at time t

Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rigid body velocity:

rigid motion: $g_b^a = \begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}$

The twist observed at the spatial frame:

$$\hat{\xi}_{ab}^s = \dot{g}_b^a {g_b^a}^{-1} = \begin{bmatrix} \dot{R}_b^a {R_b^a}^{-1} & -\dot{R}_b^a {R_b^a}^{-1} p_b^a + \dot{p}_b^a \\ 0 & 0 \end{bmatrix}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rigid body velocity:

rigid motion: $g_b^a = \begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}$

The twist observed at the spatial frame:

$$\hat{\xi}_{ab}^s = \dot{g}_b^a {g_b^a}^{-1} = \begin{bmatrix} \boxed{\dot{R}_b^a {R_b^a}^{-1}} & \boxed{-\dot{R}_b^a {R_b^a}^{-1} p_b^a + \dot{p}_b^a} \\ 0 & 0 \end{bmatrix}$$

$\hat{\omega}_{ab}^s$

v_{ab}^s: not the velocity of the body frame, but the instantaneous velocity of the point on an infinitely large body currently at the origin of the spatial (A) frame

Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rigid body velocity:

  rigid motion: $g_b^a = \begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}$

  The twist observed at the body frame:

$$\hat{\xi}_{ab}^b = g_b^{a^{-1}} \dot{g}_b^a = \begin{bmatrix} R_b^{a^{-1}} \dot{R}_b^a & R_b^{a^{-1}} \dot{p}_b^a \\ 0 & 0 \end{bmatrix}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- Rigid body velocity:

  rigid motion: $g_b^a = \begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix}$

  The twist observed at the body frame:

$$\hat{\xi}_{ab}^b = g_b^{a\,-1} \dot{g}_b^a = \begin{bmatrix} \boxed{R_b^{a\,-1} \dot{R}_b^a} & \boxed{R_b^{a\,-1} \dot{p}_b^a} \\ 0 & 0 \end{bmatrix}$$

$\hat{\omega}_{ab}^b$

v_{ab}^b: the instantaneous velocity at the body frame

Association of Velocities of a Rigid Body at Different Coordinate Frames

- The twist observed at the body frame: $\hat{\xi}_{ab}^{b}$
- The twist observed at the spatial frame: $\hat{\xi}_{ab}^{s}$
- The association of twist between these two frames: $\hat{\xi}_{ab}^{s} = g_b^a \hat{\xi}_{ab}^{b} {g_b^a}^{-1}$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- The twist observed at the body frame: $\hat{\xi}_{ab}^{b}$
- The twist observed at the spatial frame: $\hat{\xi}_{ab}^{s}$
- The association of twist between these two frames: $\hat{\xi}_{ab}^{s} = g_b^a \hat{\xi}_{ab}^{b} {g_b^a}^{-1}$

$$\Rightarrow \begin{bmatrix} \widehat{\omega}_{ab}^{s} & v_{ab}^{s} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_b^a & p_b^a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\omega}_{ab}^{b} & v_{ab}^{b} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} {R_b^a}^{-1} & -{R_b^a}^{-1} p_b^a \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \widehat{\omega}_{ab}^{s} & v_{ab}^{s} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1} & -R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1} p_b^a + R_b^a v_{ab}^{b} \\ 0 & 1 \end{bmatrix}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- We have:

$$\widehat{\omega}_{ab}^{s} = R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1}$$

$$v_{ab}^{s} = -R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1} p_b^a + R_b^a v_{ab}^{b}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- First, we know given any $u \in \mathbb{R}^3$ and $\mathbf{R} \in SO(3)$, the following always holds:

$$R\hat{u}R^{-1} = \widehat{Ru}$$

Proof. Letting $r_i^\top$ denotes the i-th row of R

$$R\hat{u}R^{-1} = \begin{bmatrix} r_1^\top(u \times r_1) & r_1^\top(u \times r_2) & r_1^\top(u \times r_3) \\ r_2^\top(u \times r_2) & r_2^\top(u \times r_2) & r_2^\top(u \times r_3) \\ r_3^\top(u \times r_3) & r_3^\top(u \times r_2) & r_3^\top(u \times r_3) \end{bmatrix} \quad \boxed{a^\top(b \times c) = b^\top(c \times a)}$$

$$= \begin{bmatrix} 0 & -r_3^\top u & r_2^\top u \\ r_3^\top u & 0 & -r_1^\top u \\ -r_2^\top u & r_1^\top u & 0 \end{bmatrix} = \widehat{Ru}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- As a result:

$$\widehat{\omega}_{ab}^{s} = R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1} \Rightarrow \widehat{\omega}_{ab}^{s} = \widehat{R_b^a \omega_{ab}^{a}} \Rightarrow \omega_{ab}^{s} = R_b^a \omega_{ab}^{a}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- As a result:

$$\widehat{\omega}_{ab}^{s} = R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1} \Rightarrow \widehat{\omega}_{ab}^{s} = \widehat{R_b^a \omega_{ab}^a} \Rightarrow \omega_{ab}^{s} = R_b^a \omega_{ab}^{a}$$

- Also:

$$v_{ab}^{s} = -R_b^a \widehat{\omega}_{ab}^{b} {R_b^a}^{-1} p_b^a + R_b^a v_{ab}^{b} = -\widehat{\omega}_{ab}^{s} p_b^a + R_b^a v_{ab}^{b}$$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- As a result:

$$\widehat{\omega}_{ab}^{s} = R_b^a \widehat{\omega}_{ab}^{b} R_b^{a^{-1}} \Rightarrow \widehat{\omega}_{ab}^{s} = \widehat{R_b^a \omega_{ab}^a} \Rightarrow \omega_{ab}^{s} = R_b^a \omega_{ab}^a$$

- Also:

$$v_{ab}^{s} = -R_b^a \widehat{\omega}_{ab}^{b} R_b^{a^{-1}} p_b^a + R_b^a v_{ab}^b = -\widehat{\omega}_{ab}^{s} p_b^a + R_b^a v_{ab}^b$$
$$= \hat{p}_{ab}^{s} \omega_b^a + R_b^a v_{ab}^b = \hat{p}_{ab}^{s} R_b^a \omega_{ab}^a + R_b^a v_{ab}^b$$

$a \times b$
$= -b \times a$
$\Rightarrow \hat{a} b = -\hat{b} a$

Association of Velocities of a Rigid Body at Different Coordinate Frames

- We have:

$$\omega_{ab}^{s} = R_b^a \omega_{ab}^{a}$$
$$v_{ab}^{s} = \hat{p}_{ab}^{s} R_b^a \omega_{ab}^{a} + R_b^a v_{ab}^{b}$$

- Rewrite in a matrix form:

$$\begin{bmatrix} \omega_{ab}^{s} \\ v_{ab}^{s} \end{bmatrix} = \underbrace{\begin{bmatrix} R_b^a & 0 \\ \hat{p}_{ab}^{s} R_b^a & R_b^a \end{bmatrix}}_{\mathbf{Ad}_g\text{: adjoint transformation for } g_{ab}} \begin{bmatrix} \omega_{ab}^{b} \\ v_{ab}^{b} \end{bmatrix}$$

For any $g \in SE(3)$ and $\hat{\xi} \in se(3)$, $g\hat{\xi}g^{-1} \in se(3)$ is a twist with twist coordinate $\mathbf{Ad}_g \xi$

- We have $\hat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3)$ with the twist coordinate $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix}$
- We have:

$$g\hat{\xi}g^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix}$$

For any $g \in SE(3)$ and $\hat{\xi} \in se(3)$, $g\hat{\xi}g^{-1} \in se(3)$ is a twist with twist coordinate $\mathbf{Ad}_g \xi$

- We have:

$$
\begin{aligned}
g\hat{\xi}g^{-1} &= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R\hat{\omega} & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R\hat{\omega}R^{-1} & -R\hat{\omega}R^{-1}p + Rv \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

For any $g \in SE(3)$ and $\hat{\xi} \in se(3)$, $g\hat{\xi}g^{-1} \in se(3)$ is a twist with twist coordinate $\mathbf{Ad}_g \xi$

- We have:

$$
\begin{aligned}
g\hat{\xi}g^{-1} &= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R\widehat{\omega} & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R\widehat{\omega}R^{-1} & -R\widehat{\omega}R^{-1}p + Rv \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \widehat{R\omega} & -\widehat{R\omega}p + Rv \\ 0 & 0 \end{bmatrix} \in se(3)
\end{aligned}
$$

For any $g \in SE(3)$ and $\hat{\xi} \in se(3)$, $g\hat{\xi}g^{-1} \in se(3)$ is a twist with twist coordinate $\mathbf{Ad}_g\xi$

- We have $\hat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3)$ with the twist coordinate $\xi = \begin{bmatrix} \omega \\ v \end{bmatrix}$
- We have: $g\hat{\xi}g^{-1} = \begin{bmatrix} \widehat{R\omega} & -\widehat{R\omega}p + Rv \\ 0 & 0 \end{bmatrix} \in se(3)$
- The twist coordinate:

$$\begin{bmatrix} R\omega \\ -\widehat{R\omega}p + Rv \end{bmatrix} = \begin{bmatrix} R\omega \\ \hat{p}R\omega + Rv \end{bmatrix} = \begin{bmatrix} R & 0 \\ \hat{p}R & R \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} = \mathrm{Ad}_g\xi$$

Previously, We Discussed Inverse Kinematic

- Connecting a twist and joint angles in linear forms:

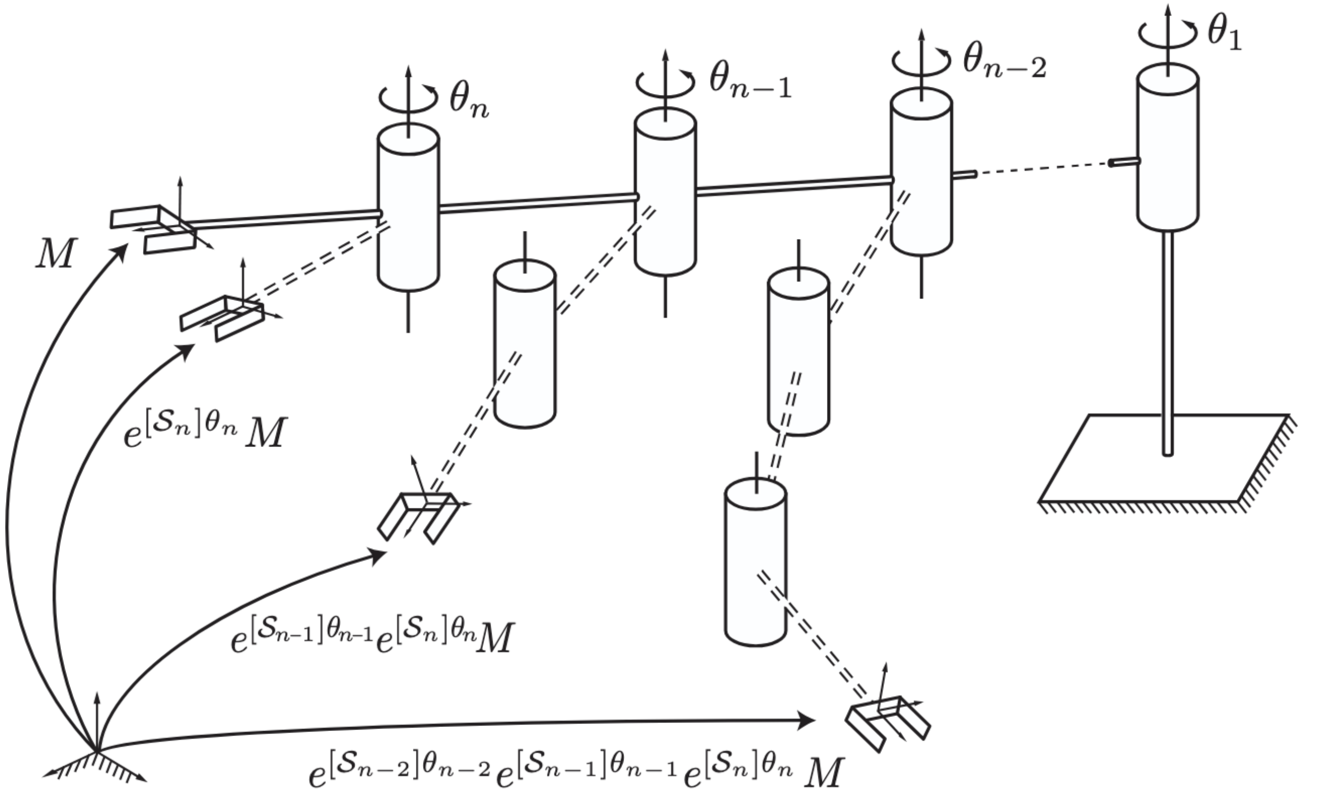
$$\xi_S = \underbrace{\mathcal{S}_1}_{J_{s1}} \dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}} \dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}} \dot{\theta}_3 + \cdots$$

$$\begin{aligned} \xi_S &= \begin{bmatrix} J_{s1} & J_{s2}(\theta) & \cdots & J_{sn}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \\ &= J_s(\theta)\dot{\theta}. \end{aligned}$$

Product of Exponential for an N-Link Open Chain

- Spatial transformation of the open chain:

$$T(\theta_1, \theta_2, \dots, \theta_n) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \dots e^{\hat{\xi}_n\theta_n} M$$



Product of Exponential for an N-Link Open Chain

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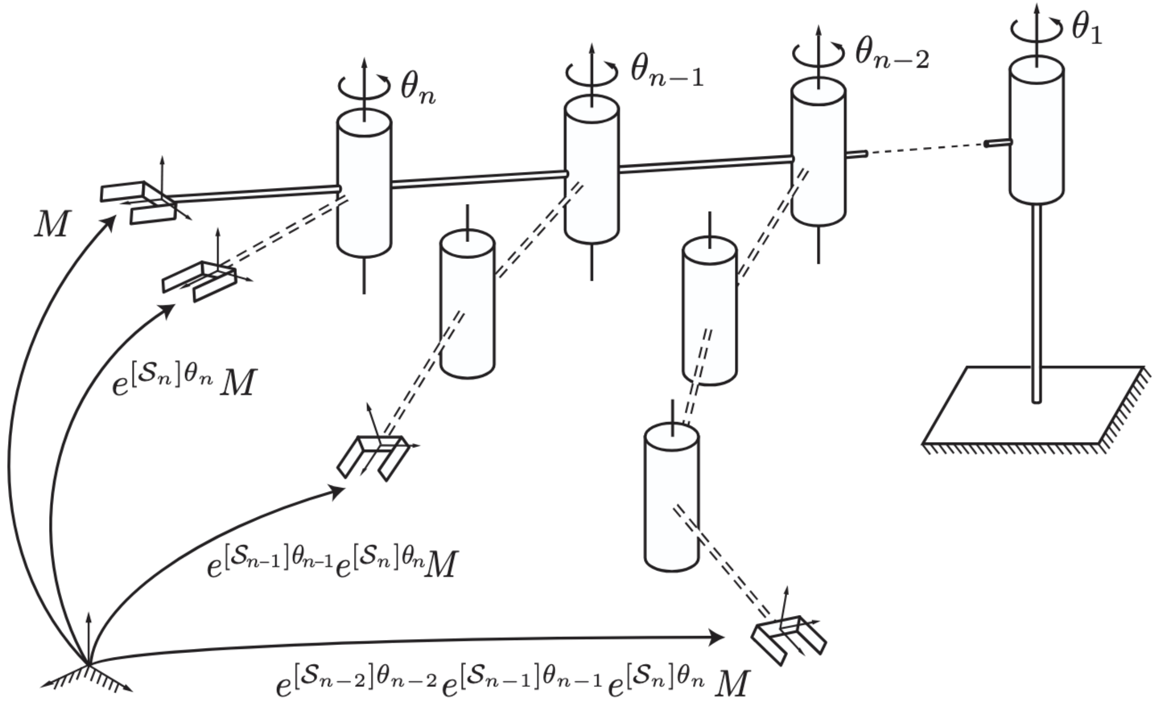
- The twist of the spatial transformation (check page 78):

$$\hat{\xi}^s = \dot{T}T^{-1}$$



Product of Exponential for an N-Link Open Chain

- The inverse of the spatial transformation:

$$T^{-1} = M^{-1} e^{-\hat{\xi}_n \theta_n} \dots e^{-\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1}$$

- The derivative of the transformation:

$$\begin{aligned} \dot{T} &= \left(\frac{d e^{\hat{\xi}_1 \theta_1}}{dt} \right) e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} M + e^{\hat{\xi}_1 \theta_1} \left(\frac{d e^{\hat{\xi}_2 \theta_2}}{dt} \right) \dots e^{\hat{\xi}_n \theta_n} M + \cdots \\ &= \hat{\xi}_1 \dot{\theta}_1 e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} M + e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 \dot{\theta}_2 e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} M + \cdots \end{aligned}$$

Product of Exponential for an N-Link Open Chain

- The inverse of the spatial transformation:

$$T^{-1} = M^{-1} e^{-\hat{\xi}_n \theta_n} \dots e^{-\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1}$$

- The derivative of the transformation:

$$\begin{aligned}\dot{T} &= \left(\frac{de^{\hat{\xi}_1 \theta_1}}{dt}\right) e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} M + e^{\hat{\xi}_1 \theta_1} \left(\frac{de^{\hat{\xi}_2 \theta_2}}{dt}\right) \dots e^{\hat{\xi}_n \theta_n} M + \cdots \\ &= \hat{\xi}_1 \dot{\theta}_1 e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} M + e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 \dot{\theta}_2 e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} M + \cdots\end{aligned}$$

- The twist of the spatial transformation (check page 78):

$$\hat{\xi}^s = \dot{T} T^{-1} = \hat{\xi}_1 \dot{\theta}_1 + e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 \dot{\theta}_2 e^{-\hat{\xi}_1 \theta_1} + e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \hat{\xi}_3 \dot{\theta}_3 e^{-\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1} + \cdots$$

Product of Exponential for an N-Link Open Chain

- The twist of the spatial transformation:

$$\hat{\xi}^s = \dot{T}T^{-1} = \hat{\xi}_1\dot{\theta}_1 + e^{\hat{\xi}_1\theta_1}\hat{\xi}_2\dot{\theta}_2 e^{-\hat{\xi}_1\theta_1} + e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\hat{\xi}_3\dot{\theta}_3 e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1} + \cdots$$

$$= \hat{\xi}_1\dot{\theta}_1 + e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{-\hat{\xi}_1\theta_1}\dot{\theta}_2 + e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\hat{\xi}_3 e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}\dot{\theta}_3 + \cdots$$

For any $g \in SE(3)$ and $\hat{\xi} \in se(3)$, $g\hat{\xi}g^{-1} \in se(3)$ is a twist with twist coordinate $\mathbf{Ad}_g\xi$

Product of Exponential for an N-Link Open Chain

- The twist of the spatial transformation:

$$\hat{\xi}^s = \dot{T}T^{-1} = \hat{\xi}_1\dot{\theta}_1 + e^{\hat{\xi}_1\theta_1}\hat{\xi}_2\dot{\theta}_2 e^{-\hat{\xi}_1\theta_1} + e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\hat{\xi}_3\dot{\theta}_3 e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1} + \cdots$$
$$= \hat{\xi}_1\dot{\theta}_1 + e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{-\hat{\xi}_1\theta_1}\dot{\theta}_2 + e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\hat{\xi}_3 e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}\dot{\theta}_3 + \cdots$$

For any $g \in SE(3)$ and $\hat{\xi} \in se(3)$, $g\hat{\xi}g^{-1} \in se(3)$ is a twist with twist coordinate $\mathbf{Ad}_g\xi$

- The twist coordinate:

$$\xi^s = \underbrace{\xi_1}_{\mathrm{J}_{\xi_1}(\theta)}\dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}}\xi_2}_{\mathrm{J}_{\xi_2}(\theta)}\dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}}\hat{\xi}_3}_{\mathrm{J}_{\xi_3}(\theta)}\dot{\theta}_3 + \cdots$$

Product of Exponential for an N-Link Open Chain

- The twist coordinate:

$$\xi^s = \mathrm{J}_{\xi_1}(\theta)\dot{\theta}_1 + \mathrm{J}_{\xi_2}(\theta)\dot{\theta}_2 + \mathrm{J}_{\xi_3}(\theta)\dot{\theta}_3 + \cdots$$

$$= [\mathrm{J}_{\xi_1} \quad \mathrm{J}_{\xi_2} \quad \mathrm{J}_{\xi_3} \quad \ldots] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \vdots \end{bmatrix} = \mathbf{J}_{\boldsymbol{\xi}}\dot{\boldsymbol{\theta}}$$

Inverse Kinematic

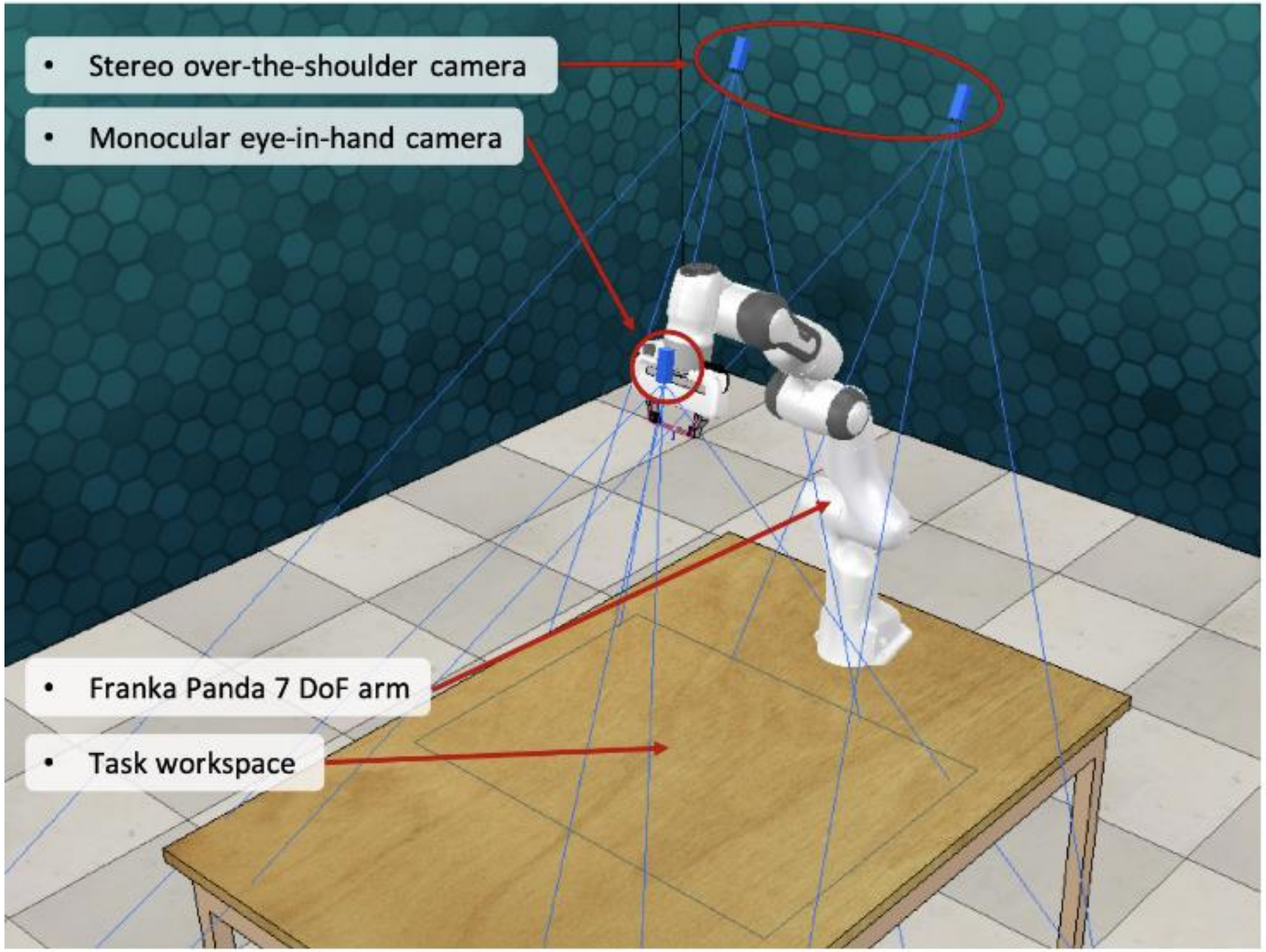
- Newton-Raphson Method:

$$\Delta\theta = J^{-1}(\theta^0)\left(x_d - f(\theta^0)\right).$$

$$\Downarrow$$

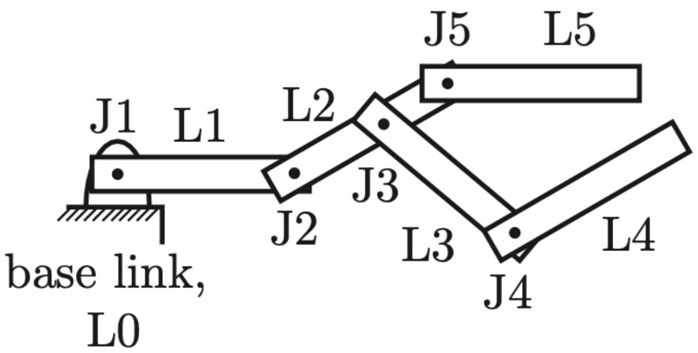
$$\Delta\theta = \boldsymbol{J}_{\xi}^{-1}\left(\xi^{\Delta}\right), \text{where } \xi^{\Delta} = \log\left(T_{current}^{-1}(\theta)T_{target}\right)$$

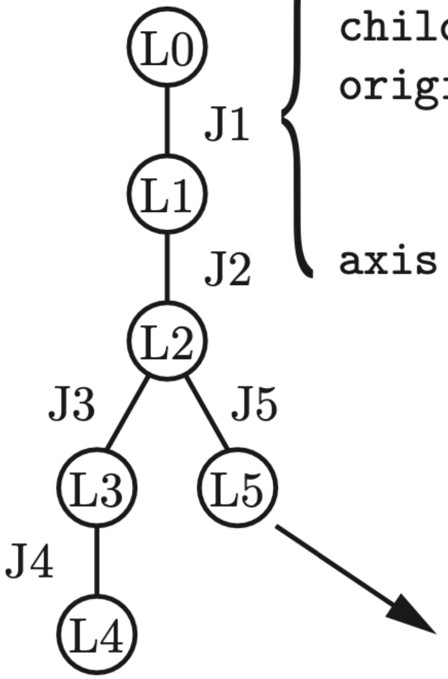
How to Build a Robot in the Simulator?



RLBench: The Robot Learning Benchmark & Learning Environment. James et al.

Represent a Robot as a Tree





parent:	J1's parent link, L0
child:	J1's child link, L1
origin:	the x–y–z and roll–pitch–yaw coords of the L1 frame relative to the L0 frame when J1 is zero
axis:	the x–y–z unit vector along the rotation axis in the L1 frame

mass:	L5's mass
origin:	the x–y–z and roll–pitch–yaw coords of a frame at the center of mass of L5, relative to the L5 frame
inertia:	six unique entries of inertia matrix in the **origin** frame

The Robot’s URDF File

```
<?xml version="1.0" ?>
<robot name="panda">
  <material name="aluminum">
    <color rgba="0.5 0.5 0.5 1"/>
  </material>
  <link name="panda_link0">
    <visual>
      <geometry>
        <mesh
filename="franka_description/meshes/visual/link0.glb"/>
      </geometry>
    </visual>
    <collision>
      <geometry>
        <mesh
filename="franka_description/meshes/collision/link0.stl"/>
      </geometry>
    </collision>
    <inertial>
      <origin rpy="0 0 0" xyz="-0.041018 -0.00014 0.049974"/>
      <mass value="0.629769"/>
      <inertia ixx="0.00315" ixy="8.2904e-07" ixz="0.00015"
iyy="0.00388" iyz="8.2299e-06" izz="0.004285"/>
    </inertial>
  </link>
```

The mesh of the link for visualization

The mesh of the link for collision checking

Physical properties of the link

```
  <link name="panda_link1">
    <visual>
      <geometry>
        <mesh
filename="franka_description/meshes/visual/link1.glb"/>
      </geometry>
    </visual>
    <collision>
      <geometry>
        <mesh
filename="franka_description/meshes/collision/link1.stl"/>
      </geometry>
    </collision>
    <inertial>
      <origin rpy="0 0 0" xyz="0.003875 0.002081 -0.04762"/>
      <mass value="4.970684"/>
      <inertia ixx="0.70337" ixy="-0.000139" ixz="0.006772"
iyy="0.70661" iyz="0.019169" izz="0.009117"/>
    </inertial>
  </link>
  <joint name="panda_joint1" type="revolute">
    <safety_controller k_position="100.0" k_velocity="40.0"
soft_lower_limit="-2.8973" soft_upper_limit="2.8973"/>
    <origin rpy="0 0 0" xyz="0 0 0.333"/>
    <parent link="panda_link0"/>
    <child link="panda_link1"/>
    <axis xyz="0 0 1"/>
    <limit effort="87" lower="-2.8973" upper="2.8973"
velocity="2.1750"/>
    <dynamics D="1" K="7000" damping="0.003" friction="0.0"
mu_coulomb="0" mu_viscous="16"/>
  </joint>
```